1 Using this Document and Disclaimer

This document is intended to record the solutions to the 2016 comprehensive exam as I understand them. No guarantees are made about the correctness of these solutions, though I have tried to careful to do things correctly. Throughout this document, I have also made an effort to point out how the reader might remember the details common formulae. It is important to note that these explanations are *not* intended to be rigorous justifications, but rather are intended as a collection of mnemonics from which one might hope to interpolate the correct formula from an imperfect memory.

With this said, I do hope this document will prove as useful to others as I hope it will be for myself in creating it. Feedback is appreciated, and should be directed to the author's email: myersr(at_the_system)physics.ucla.edu.

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2 Problem 1: Classical Mechanics

2.1 Problem Statement (Pendulum in a Rotating Frame/Lagrange Multipliers)

Consider a pendulum made of a very heavy mass m suspended by a light wire of length ℓ from a tall ceiling. The pendulum is allowed to swing freely for long periods of time, and moves in both the east-west and north-south directions.

(a) Choose a set of axes such that x is east and y is north and z is vertically upwards and assume small oscillations to calculate the two coupled differential equations for the motion of the pendulum in a frame rotating with angular velocity $\Omega||z$, much smaller than the pendulum characteristic frequency so you can neglect all effects $\propto \Omega^2$.

(b) If the pendulum starts oscillating in the x-plane with zero initial velocity and initial offset x_0, use perturbation theory to obtain the first order correction to the trajectory.

(c) Find the full solution using the substitution $\eta = x + iy$ and then trying a solution for the resulting equation of motion of the form $\eta(t) = f(t)e^{-i\Omega t}$.

2.2 Part (a)

The problem statement is an exceedingly long run-on sentence, but if we parse carefully, we note that the problem only asks us to find the equations of motion, not solve them. Furthermore, if we look at parts (b) and (c), it seems we are expected to deal with these equations in terms of Cartesian coordinates. So, while it may be tempting to convert to spherical coordinates since that's the symmetry of the problem, we will actually be better off staying in Cartesian coordinates over the course of this problem. This does add a bit of a complication in imposing the constraint that the pendulum have length ℓ since we can no longer impose this by fixing the radial coordinate in the spherical coordinate system. Instead, we need to impose the constraint via a Lagrange multiplier in our Lagrangian.

Suppose first that the coordinate system $\mathbf{r}' = (\mathbf{x}', \mathbf{y}', \mathbf{z}')$ is the fixed, stationary frame of reference and that $\mathbf{r} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the rotating frame. If $R(\theta)$ is the rotation matrix for a rotation about the z-axis by an angle θ, then $\mathbf{r} = R(\Omega t)\mathbf{r}'$. Since a rotation about the z-axis takes the general form

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \exp\left[-i(\sigma_y \oplus 0)\Omega t\right], \tag{2.2.1}$$

where σ_y is the Pauli y matrix. It therefore follows that

$$\dot{R}(\Omega t) = -i\Omega(\sigma_y \oplus 0)R(\Omega t) = \Omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R(\Omega t) = \Omega \begin{bmatrix} -\sin\Omega t & -\cos\Omega t & 0 \\ \cos\Omega t & -\sin\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.2.2}$$

The reason we need to know about this rotation operator is because will need to convert the kinetic energy $\frac{1}{2}m\dot{\mathbf{r}}'$ to be in terms of the rotating coordinates. Writing the relationship $\mathbf{r}' = R(-\Omega t)\mathbf{r}$, we are free to now compute the time derivative

$$\dot{\mathbf{r}}' = R(-\Omega t)\dot{\mathbf{r}} + \dot{R}(-\Omega t)\mathbf{r} = R(-\Omega t)\dot{\mathbf{r}} + i\Omega(\sigma_y \oplus 0)R(-\Omega t). \tag{2.2.3}$$

From here, we are free to compute $\dot{\mathbf{r}}' \cdot \dot{\mathbf{r}}'$ to be

$$\dot{\mathbf{r}}'^{\,2} = \dot{\mathbf{r}}^2 + \Omega^2\mathbf{r}^2 + i\Omega\left[\dot{\mathbf{r}}^T(\sigma_y \oplus 0)\mathbf{r} - \mathbf{r}^T(\sigma_y \oplus 0)\dot{\mathbf{r}}\right], \tag{2.2.4}$$

where we have used $R^TR = 1$, $[R, \sigma_y \oplus 0] = 0$, and $\sigma_y^T = -\sigma_y$. The commutator can be evaluated directly, but it is easier to note that R is a matrix exponential of $\sigma_y \oplus 0$, and so they necessarily commute. Next, note that for any vectors $\mathbf{a}$,$\mathbf{b}$, and matrix A, it is necessarily the case that

$$\mathbf{a}^TA\mathbf{b} = \left(\mathbf{a}^TA\mathbf{b}\right)^T = \mathbf{b}^TA^T\mathbf{a}. \tag{2.2.5}$$

In our case, we are interested in making the two remaining bilinear forms in (2.2.4) have the same vector on the same side, so we might be able to combine them in some way and simplify the expression. Conveniently, this identity implies $\dot{\mathbf{r}}^T(\sigma_y \oplus 0)\mathbf{r} = -\mathbf{r}^T(\sigma_y \oplus 0)\dot{\mathbf{r}}$ by the previously noted identity $\sigma_y^T = -\sigma_y$. Therefore, we have

$$\dot{\mathbf{r}}'^{\,2} = \dot{\mathbf{r}}^2 + \Omega^2\mathbf{r}^2 - 2i\Omega\mathbf{r}^T(\sigma_y \oplus 0)\dot{\mathbf{r}} = \dot{\mathbf{r}}^2 + \Omega\mathbf{r}^2 - 2\Omega\left[y\dot{x} - x\dot{y}\right]. \tag{2.2.6}$$

Now, to impose the constraint $\mathbf{r}^2 = \ell^2$, we introduce the Lagrange multiplier α. Our Lagrangian is now

$$L = \frac{1}{2}m\left(\dot{\mathbf{r}}^2 + \Omega^2\mathbf{r}^2 - 2\Omega[y\dot{x} - x\dot{y}]\right) - mgz + \frac{1}{2}\alpha\left(\mathbf{r}^2 - \ell^2\right). \tag{2.2.7}$$

Hence, the equations of motion for the z coordinate is

$$m\ddot{z} = m\Omega^2 z - mg + \alpha z = -mg + (m\Omega^2 + \alpha)z \approx -mg + \alpha z, \tag{2.2.8}$$

where we have dropped the term of order $\mathcal{O}(\omega^2)$ as suggested by the problem statement. The remaining two equations of motion are

$$\begin{aligned} m\ddot{x} - 2m\Omega\dot{y} &= \alpha x, \\ m\ddot{y} + 2m\Omega\dot{x} &= \alpha y. \end{aligned} \tag{2.2.9}$$

If we now take two derivatives of the constraint, we find the condition

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{2.2.10}$$

This becomes useful to us if we multiply the x equation by x, the y equation by y, and similarly for z, and then sum the results,

$$-mgz+\alpha\ell^2 = m(x\ddot{x}+y\ddot{y}+z\ddot{z})-2m\Omega\left[x\dot{y} - y\dot{x}\right] = -m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-2m\Omega\left[x\dot{y} - y\dot{x}\right]. \tag{2.2.11}$$

Hence,

$$\alpha\ell^2 = mgz - m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - 2\Omega\left[x\dot{y} - y\dot{x}\right]. \tag{2.2.12}$$

However, since[1] $z = -\sqrt{\ell^2 - (x^2 + y^2)}$, we can write

$$\dot{z}^2 = \frac{(x\dot{x} + y\dot{y})^2}{z^2}, \tag{2.2.13}$$

which would then allow us to eliminate the z dependence from α, and using the resulting expression for α, we would be left with two equations of motion for the system, (2.2.9).

For small oscillations, inspection of (2.2.9) shows that only zeroth order terms in α are capable of contributing to the motion. The only term that is capable of making a zeroth order contribution is the term mgz since $z = -\ell\sqrt{1 - (x^2 + y^2)/\ell^2} \approx -\ell$. Hence, $\alpha \approx= -mg/\ell$ which then leads us to the equations

$$\begin{aligned} m\ddot{x} - 2m\Omega\dot{y} &= -\frac{mg}{\ell}x, \\ m\ddot{y} + 2m\Omega\dot{x} &= -\frac{mg}{\ell}y, \end{aligned} \tag{2.2.14}$$

for small oscillations.

[1] We must choose the negative branch to make the bottom of the pendulum, which is already a distance ℓ away, be a distance ℓ away, downwards.

2.3 Part (b)

For the classical perturbation, we expand the exact solution in powers of Ω, since we are told in the problem statement that this is supposed to be a small parameter. To first order, the solution is then $x(t) = x^{(0)}(t) + \Omega x^{(1)}(t) + \mathcal{O}(\Omega^2)$ and similarly for $y(t)$. The equations of motion are then

$$\begin{aligned} 0 &= \left(m\ddot{x}^{(0)} + \frac{mg}{\ell}x^{(0)}\right) + \Omega\left(m\ddot{x}^{(1)} + \frac{mg}{\ell}x^{(1)} - 2m\dot{y}^{(0)}\right) + \mathcal{O}(\Omega^2), \\ 0 &= \left(m\ddot{y}^{(0)} + \frac{mg}{\ell}y^{(0)}\right) + \Omega\left(m\ddot{y}^{(1)} + \frac{mg}{\ell}y^{(1)} + 2m\dot{x}^{(0)}\right) + \mathcal{O}(\Omega^2), \end{aligned} \tag{2.3.1}$$

which we now demand holds order by order in Ω. The zeroth order correction is then clearly a harmonic oscillator, whose solution we know to be

$$x^{(0)}(t) = x_0 \cos\omega_0 t, \qquad y^{(0)}(t) = 0, \tag{2.3.2}$$

for the given boundary conditions and where $\omega_0 = \sqrt{g/\ell}$. The first order corrections then satisfy

$$\begin{aligned} 0 &= \ddot{x}^{(1)} + \omega_0^2 x^{(1)}, \\ 0 &= \ddot{y}^{(1)} + \omega_0^2 y^{(1)} - 2\omega_0 x_0 \sin\omega_0 t. \end{aligned} \tag{2.3.3}$$

The equation for $x^{(1)}(t)$ is again that of a harmonic oscillator, but since we forced the zeroth order solution to satisfy the boundary conditions, the first order correction must contribute zero initial position and velocity. Therefore, $x^{(1)}(t) = 0$. The equation for $y^{(1)}(t)$ is the equation of a driven harmonic oscillator with no damping, so we know the particular solution will look like a cosine function[2] of the same frequency, ω_0. But if we were to do this, we would find no solution. We could think of this issue as being on-resonance with zero damping. The solution is therefore of the form $At\cos\omega_0 t$, which implies $A = -x_0$ so the particular solution is $y_p^{(1)}(t) = -x_0 t\cos\omega_0 t$.

Since we already know the homogeneous solution is $y_h^{(1)}(t) = A\sin\omega_0 t + B\cos\omega_0 t$, we just need to impose the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$. These imply $B = 0$ and $A = x_0/\omega_0$. Hence, the full first order solution to (2.2.14) is

$$\begin{aligned} x(t) &= x_0\cos\omega_0 t + \mathcal{O}(\Omega^2), \\ y(t) &= \Omega\frac{x_0}{\omega_0}\sin\omega_0 t - x_0\Omega t\cos\omega_0 t + \mathcal{O}(\Omega^2). \end{aligned} \tag{2.3.4}$$

We should also point out that the prefactor of t represents a singularity in the perturbation, making this perturbation valid only for small t.

[2]The proper way of doing this without already knowing the answer would be to Fourier transform $y^{(1)}$.

2.4 Part (c)

Before we begin, it's worth pointing out that we might have guessed that this coordinate transformation would work by noting that the equations of motion are essentially those of a charged particle in a magnetic field with the addition of a harmonic potential. In any case, we are going to make the variable transformation by multiplying the y equation of (2.2.14) by i and then adding the two equations of motion to find

$$0 = \ddot{\eta} + \omega_0^2 \eta + 2\Omega(i\dot{x} - \dot{y}) = \ddot{\eta} + 2i\Omega\dot{\eta} + \omega_0^2 \eta, \tag{2.4.1}$$

which has the form of a damped harmonic oscillator. The suggested form of solution, however, gives a nice cancellation:

$$\ddot{f} = -(\Omega^2 + \omega_0^2) f. \tag{2.4.2}$$

While we could solve this problem exactly, we are still ignoring terms of order Ω^2, so the solution is again sines and cosines in ω_0. If it were the case that we were able to find a solution for f which was real, then $f(0) = x(0)$, but we would also have $\dot{\eta}(0) = \dot{f}(0) - i\Omega f(0)$. This would imply $\dot{x}(0) = \dot{f}(0)$, but would also imply $\dot{y}(0) = -\Omega f(0) = -\Omega x(0)$. Since $\dot{y}(0) = 0$ but $\dot{x}(0) \neq 0$, it is impossible to find a real-valued function $f(t)$.

Instead, we must write $f(t) = Ae^{-i\omega_0 t} + Be^{i\omega_0 t}$ for some $A \in \mathbb{C}$ and we must satisfy the initial conditions $f(0) = x_0$ and $0 = \dot{f}(0) - i\Omega f(0)$. The first of these implies $A + B = x_0$ while the latter implies $-i\omega_0 A + i\omega_0 B = -i\Omega x_0$, or $A - B = \frac{\Omega}{\omega_0} x_0$. Therefore,

$$A = x_0 \frac{\omega_0 + \Omega}{2\omega_0}, \qquad B = x_0 \frac{\omega_0 - \Omega}{2\omega_0}. \tag{2.4.3}$$

This implies

$$f(t) = x_0 \cos\omega_0 t - i x_0 \frac{\Omega}{\omega_0} \sin\omega_0 t, \tag{2.4.4}$$

which then allows us to write the full solution for x and y as

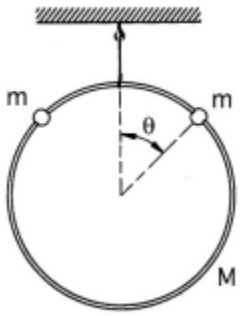
$$\begin{aligned} x(t) &= x_0 \left(\cos\omega_t \cos\Omega t - \frac{\Omega}{\omega_0} \sin\omega_0 \sin\Omega t \right), \\ y(t) &= x_0 \left(\cos\omega_0 t \sin\Omega t - \frac{\Omega}{\omega_0} \sin\omega_0 t \cos\Omega t \right). \end{aligned} \tag{2.4.5}$$

If we look carefully, we can even note that the perturbation we found in part (b) is indeed the expansion of the exact solution found here to first order in Ω.

3 Problem 2: Classical Mechanics

3.1 Problem Statement (Newtonian Mechanics or Lagrange Multipliers)

A ring of mass $M = 0.1$ kg hangs from a thread, and two beads of mass $m = 0.2$ kg slide on it without friction. The beads are released simultaneously from rest at the top of the ring and slide down opposite sides. The ring is initially motionless, but when the beads pass a critical angle, θ_c, the ring is observed to start moving upwards. Find the value of θ_c.



3.2 Solution (Newtonian Mechanics)

To analyze this problem with Newton's equations, we note that there will be a normal force due to the hoop which acts on the beads to keep them on the hoop. Since the beads must act on the hoop in an identical but opposite fashion, there will be a force on the hoop due to the two beads in addition to the force supplied by gravity and the tension in the thread. When the vertical components of these normal forces on the hoop are capable of matching the gravitational force on the hoop, the tension in the thread will be zero and any additional force from the beads will cause it to rise. This is the condition we will use to determine θ_c. First, however, we must compute these forces.

To get the normal force[3] acting on one of the beads, we note via sufficient trigonometry that the sum of forces on the bead which point towards the center of the hoop is $F_c = mg\cos\theta - N$ where θ is the angular position of the bead with $\theta = 0$ at the top of the hoop. But since the bead travels on a circular path, it must be the case that $F_c = mv^2/r$, so $N = m(g\cos\theta - v^2/r)$. To find v^2/r, we can consider the energy of the bead. Since the bead and hoop do no work on each other, the energy of the bead will be conserved. It

[3]We note that it doesn't matter which direction we assume N to point, as long as we are consistent in our convention.

then follows that if we choose the top of the hoop to be our level of zero potential, that $0 = -mgr(1-\cos\theta) + \frac{1}{2}mv^2$ from which we find $v^2/r = 2mg(1-\cos\theta)$.

We now have $N = mg(3\cos\theta - 2)$. The condition for the hoop to jump is then written $0 = 2N\cos\theta + Mg$, as discussed above. This may now be written in the form

$$6\cos^2\theta - 4\cos\theta + \frac{M}{m} = 0. \tag{3.2.1}$$

This may be solved for $\cos\theta$ to find

$$\cos\theta_c = \frac{4 \pm \sqrt{16 - 24M/m}}{12} = \frac{2 \pm \sqrt{4 - 6M/m}}{6} = \frac{2 \pm 1}{6} = \begin{cases} 1/6 \\ 1/2 \end{cases} \tag{3.2.2}$$

Since $\cos\theta$ is a decreasing function on $[0, \pi/2]$, as θ increases, the value of $\cos\theta$ will be $1/2$ before it is $1/6$. Therefore, we accept the solution $\cos\theta_c = 1/2$, which then implies $\theta_c = 60°$.

3.3 Solution (Lagrange Multipliers)

The solution using Newtonian mechanics given above is definitely the faster of these two options if you think to go that path. However, if nothing but for the sake of practice, we will present a solution using Lagrange multipliers as well. Well, as we will see, once we've seen how to do the problem from Newton's laws, the Lagrangian calculation will just trade trigonometry for calculus.

We will model the hoop as static[4], and then compute the vertical component of the normal force from the constraint, and require it to be equal to the gravitational force acting on the hoop. In fact, since the two beads don't interact, we will calculate the force supplied by only one bead, and then double it. The constraint for this problem is then clearly $x^2 + y^2 = r^2$ where r is the radius of the hoop. The Lagrangian for the system is then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mg(r - y) - \frac{1}{2}\alpha(x^2 + y^2 - r^2). \tag{3.3.1}$$

The equations of motion are therefore

$$x^2 + y^2 = r^2, \quad m\ddot{x} = -\alpha x, \quad m\ddot{y} = -mg - \alpha y. \tag{3.3.2}$$

The constraint force in the y-direction on the bead is therefore $-\alpha y$. Hence, the hoop experiences a force $2\alpha y$, which must match Mg. So, we must find α. Disintegrating the

[4]If we did not do this, then the tension would act as a non-holonomic constraint.

constraint twice, we find $(x\ddot{x} + y\ddot{y}) = -(\dot{x}^2 + \dot{y}^2)$. If we multiply the x and y equations of motion by x and y, respectively, and then sum the results, we find

$$m(\dot{x}^2 + \dot{y}^2) = mgy + \alpha r^2. \tag{3.3.3}$$

To determine the velocity squared, we are free to use the energy, which is clearly conserved since the Lagrangian is time independent[5]. This implies that $0 = m(\dot{x}^2 + \dot{y}^2) - 2mg(r - y)$, so we find

$$2mg(r - y) = mgy + \alpha r^2, \tag{3.3.4}$$

from which we find

$$\alpha = \frac{mg}{r^2}(2r - 3y). \tag{3.3.5}$$

It now follows that our constraint takes the form

$$M = \frac{m}{r^2}(2r - 3y)y = m(2 - 3\cos\theta)\cos\theta, \tag{3.3.6}$$

where we have changed coordinates to $y = r\cos\theta$ since we were asked to find the angle, not the height. From here, we find exactly the same polynomial in $\cos\theta$ that was found from the Newtonian analysis, and hence the same result follows.

4 Problem 3: Quantum Mechanics

4.1 Problem Statement (Tensor Products of Two-State Systems)

An apparatus is constructed that emits pairs of photons whose polarizations are quantum-mechanically correlated because each pair is in the state

$$|\psi\rangle = (|H_A, H_B\rangle - |V_A, V_B\rangle)/\sqrt{2} \tag{4.1.1}$$

where H (Horizontal) and V (Vertical) corresponding to orthogonal linear polarizations for the photons. Each photon from the pair is collected in a separate (polarization-maintaining) optical fiber and the output of the first fiber is sent to Alice while the output of the second is sent to Bob (corresponding to the A and B subscripts above).

Alice and Bob each have a fancy single-photon polarization detection system that will report a result of $+1$ if the measurement of the polarization finds H and -1 if it finds

[5]Note that this essentially follows from our assumption that the hoop is fixed, and the Lagrangian for the hoop and particle therefore factors.

V. Expressed in the H, V basis, each detector implements the measurement given by the operator

$$M_p = |H_p\rangle\langle H_p| - |V_p\rangle\langle V_p| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_p \tag{4.1.2}$$

where $p \in \{A, B\}$.

(a) The H and V directions of each detector may be said to point along the x and y axes of some coordinate systems that are fixed with respect to each detector. If Alice or Bob decides to rotate their detector about the $+z$-axis through some angle θ, the resulting measurement may be called $M_p(\theta)$. Find expressions for the following four measurement operators in the H, V basis: $M_A(0), M_A(\pi/4), M_B(\pi/8), M_B(3\pi/8)$.

(b) Find the expectation value of the following operator in the state $|\psi\rangle$:

$$\begin{aligned} \hat{W} \equiv & 2 \cdot \mathbb{I} - M_A(0) M_B\left(\frac{\pi}{8}\right) + M_A(0) M_B\left(\frac{3\pi}{8}\right) \\ & + M_A\left(\frac{\pi}{4}\right) M_B\left(\frac{\pi}{8}\right) + M_A\left(\frac{\pi}{4}\right) M_B\left(\frac{3\pi}{8}\right), \end{aligned} \tag{4.1.3}$$

where $\mathbb{I}$ is the identity matrix.

(c) The operator $\hat{W}$ is known as an *entanglement witness*; if $\langle \hat{W} \rangle \leq 0$, the state is incompatible with a local hidden variable theory. Now consider the effect of group velocity birefringence in the optical fibers, which would lead to a polarization-dependent time-delay. Consider the case where the photons emitted by the source (i.e. at some position in space before they are collected by the fibers) are in Gaussian temporal wavepackets of width τ given by

$$\psi_{temporal}(t) = \frac{1}{(2\pi\tau^2)^{1/4}} e^{-\frac{t^2}{4\tau^2}}. \tag{4.1.4}$$

Find the relative delay between H and V necessary to give $\langle \hat{W} \rangle > 0$ for photons created in the initial state $|\psi\rangle$. You should assume that the fibers (and therefore the birefringence-induced delays) are identical for Alice and Bob. The following identity may be useful:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\tau\sqrt{2\pi}} \exp\left[-\frac{t^2}{4\tau^2}\right] \exp\left[-\frac{(t+T)^2}{4\tau^2}\right] = \exp\left[-\frac{T^2}{8\tau^2}\right]. \tag{4.1.5}$$

4.2 Part (a)

To compute $M_p(\theta)$, we need only compute[6] $M_p(\theta) = R(\theta) M_p R^T(\theta)$, where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4.2.1}$$

is the rotation matrix fora rotation by angle θ in the plane[7]. Actually performing this computation,

$$M_p(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \tag{4.2.2}$$

The task of computing the desired matrices is now a matter of rote[8]:

$$\begin{aligned} M_A(0) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & M_A\left(\frac{\pi}{4}\right) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ M_B\left(\frac{\pi}{8}\right) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & M_B\left(\frac{3\pi}{8}\right) &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{4.2.3}$$

The subscripts A and B do not, of course, matter to the computation.

4.3 Part (b)

Though it is not indicated in the problem, we are supposed to interpret multiplications in (4.1.3) as tensor products, not matrix multiplications. This is unfortunate notation, but since the vector $|\psi\rangle$ given to us lives in a 4 dimensional space, we should be able to deduce the correct interpretation by noticing that the matrices $M_p(\theta)$ only act on a 2 dimensional space. One option for computing $\langle\psi|\hat{W}|\psi\rangle$ would be to first compute the 4×4 matrix $\hat{W}$ by carrying out the tensor products explicitly and then summing. However, this would be an extensive computation to perform by hand, and would not only take time, but would also increase the probability of error. A more efficient method would be to write the inner

[6] We can recall that the transposed rotation must go on the right since vectors $\mathbf{v}$ transform like $R\mathbf{v}$ and therefore $RM_p\mathbf{v} = RM_pR^TR\mathbf{v}$, which fixes the ordering, even if we are considering a passive as opposed to active transformation of the vector space, which would swap which rotation gets the transpose.

[7] Some notes on how to remember this matrix: we know there is a minus sign on one of the two sines. To figure out which one, we can always act the matrix on $(1, 0)$, and know by drawing a picture that the result should be $(\cos\theta, \sin\theta)$, which fixes the sign.

[8] If the reader has access to Mathematica, I highly recommend making use of the TeXForm command for procedurally generating the TeX code of matrices.

product in terms of the H, V basis[9]:

$$\begin{aligned}\langle\psi|\hat{W}|\psi\rangle =&|a|^2\langle H_A, H_B|\hat{W}|H_A, H_B\rangle + |b|^2\langle V_A, V_B|\hat{W}|V_A, V_B\rangle \\ &+ a^*b\langle H_A, H_B|\hat{W}|V_A, V_B\rangle + ab^*\langle V_A, V_B|\hat{W}|H_A, H_B\rangle.\end{aligned} \tag{4.3.1}$$

With this, we need to compute four inner products, but we are now in a basis where we can simply read off the values from the $M_p(\theta)$ matrices we computed above. The values are

$$\begin{aligned}\langle H_A, H_B|\hat{W}|H_A, H_B\rangle &= 2-\sqrt{2}, \\ \langle V_A, V_B|\hat{W}|V_A, V_B\rangle &= 2-\sqrt{2}, \\ \langle H_A, H_B|\hat{W}|V_A, V_B\rangle &= \sqrt{2}, \\ \langle V_A, V_B|\hat{W}|H_A, H_B\rangle &= \sqrt{2}.\end{aligned} \tag{4.3.2}$$

So, the inner product of interest is now

$$\langle\psi|\hat{W}|\psi\rangle = (2-\sqrt{2})(|a|^2+|b|^2) + \sqrt{2}(a^*b+ab^*). \tag{4.3.3}$$

If we assume the state to be normalized, we can reduce this further to

$$\langle\psi|\hat{W}|\psi\rangle = 2 + \sqrt{2}(2\Re[a^*b]-1). \tag{4.3.4}$$

So, for our particular case in which $a = 1/\sqrt{2}$ and $b = -1/\sqrt{2}$,

$$\langle\psi|\hat{W}|\psi\rangle = 2-2\sqrt{2} = 2(1-\sqrt{2}) < 0. \tag{4.3.5}$$

4.4 Part (c)

From the question here, it is again not completely clear what we are supposed to do to model the birefringence described in the problem. What we are supposed to figure our, however, is that we should modify the problem, which we were not asked to do, to now write

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|H_A, H_B, 0\rangle - |V_A, V_B, T\rangle). \tag{4.4.1}$$

That is, expand our Hilbert space by tensor producting a phase-delay vector into the space. Though we have written $|0\rangle$ and $|T\rangle$, the exact delay is not known. Instead, we assume

[9] To make the procedure more transparent, we write $|\psi\rangle = a|H_A, H_B\rangle + b|V_A, V_B\rangle$.

these extra states to be normalized so $\langle 0|0\rangle = \langle T|T\rangle = 1$ and the overlap is given by $\langle 0|T\rangle = \langle T|0\rangle = \exp(-T^2/8\tau^2)$.

With this, we should really redefine the $\hat{W}$ operator to now work on the correct space, but instead the problem expects us to assume that the entanglement witness acts on the new space like $\hat{W}_{new} = \hat{W} \otimes 1$. At the end of the day, the only thing to change in our computations (4.3.2) is the computation of the off-diagonal elements, so

$$\begin{aligned}
\langle H_A, H_B, 0|\hat{W}|H_A, H_B, 0\rangle &= 2 - \sqrt{2}, \\
\langle V_A, V_B, T|\hat{W}|V_A, V_B, T\rangle &= 2 - \sqrt{2}, \\
\langle H_A, H_B, 0|\hat{W}|V_A, V_B, T\rangle &= \sqrt{2}\exp\left[-\frac{T^2}{8\tau^2}\right], \\
\langle V_A, V_B, T|\hat{W}|H_A, H_B, 0\rangle &= \sqrt{2}\exp\left[-\frac{T^2}{8\tau^2}\right],
\end{aligned} \tag{4.4.2}$$

where we have dropped the subscript *new* everywhere. The inner product is now

$$\langle\psi|\hat{W}|\psi\rangle = 2 - \sqrt{2}\left(1 + \exp\left[-\frac{T^2}{8\tau^2}\right]\right). \tag{4.4.3}$$

If we now impose the constraint $\langle\hat{W}\rangle > 0$, this implies

$$T > 2\tau\sqrt{2\ln\left(\frac{1}{\sqrt{2}-1}\right)}. \tag{4.4.4}$$

5 Problem 4: Quantum Mechanics

5.1 Problem Statement (Non-Local Theories)

We consider a one-dimensional system with Hamiltonian $H = \frac{p^2}{2m} + V$ where the potential V gives an effective description of the physical system, and V is non-local. The matrix elements of V in the position basis $|x\rangle$, for real x, is given in terms of a real-valued square integrable function $u(x)$ which decreases to zero exponentially as $|x| \to \infty$,

$$\langle x|V|x'\rangle = \frac{\hbar^2}{2m}u(x)u(x'). \tag{5.1.1}$$

(a) Derive the integro-differential equation obeyed by a wave function $\psi_k(x)$ of energy E given in terms of the wave number k by $E = \hbar^2 k^2/2m$.

(b) Establish the solution to the equation obtained in part (a) above for the scattering of an incoming plane wave of wave-number k and unit amplitude in terms of a suitable Green function $G(x, x'; k)$. (The resulting solution is usually referred to as the Lippmann-Schwinger equation).

(c) Compute the reflection and transmission coefficients, respectively r_k,t_k as functions of the Fourier transform of $u(x)$.

(d) Verify that probability is conserved during the process so that $r_k^2 + t_k^2 = 1$.

5.2 Part (a)

In its coordinate-free representation, the equation of motion for a wave function is given by

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{5.2.1}$$

If we now compute the overlap of this vector with $\langle x|$ and insert a complete set of states, we find

$$i\hbar\frac{\partial}{\partial t}\langle x|\psi\rangle = \int \mathrm{d}x'\langle x|H|x'\rangle\langle x'|\psi\rangle. \tag{5.2.2}$$

Using the representation (5.1.1), this may be written

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{\hbar^2}{2m}\int \mathrm{d}x' u(x)u(x')\psi(x',t). \tag{5.2.3}$$

If we are instead interested in the time-independent equation, as we are in this problem, we do the same for $E|\psi\rangle = H|\psi\rangle$, so

$$\frac{\partial^2}{\partial x^2}\psi(x) + k^2\psi(x) = \int \mathrm{d}x' u(x)u(x')\psi(x'). \tag{5.2.4}$$

5.3 Part (b)

This part of the question is a bit misleading because we are not, in fact, supposed to find the inverse to the operator $\partial^2 + k^2 - u(x)\int \mathrm{d}x' u(x')$. Instead, we use a dirty trick and write

(5.2.4) in the form $(\partial^2 + k^2)\psi = J(x)$ for some function $J(x)$ which we will treat as a source for what is now the Klein-Gordon equation.

If we now define $G(x, x'; k)$ to be the Green's function for the Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) G(x, x'; k) = \delta(x - x'), \tag{5.3.1}$$

then by letting[10] $y = x - x'$ and writing the Green's function in terms of its Fourier transform, $G(y; k) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega y} \tilde{G}(\omega; k)$, the Klein-Gordon equation becomes

$$\int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega y}(-\omega^2 + k^2)\tilde{G}(\omega; k) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega y}, \tag{5.3.2}$$

where we have used the Fourier representation of the delta function, $\delta(y) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega y}$. Since the Fourier complex exponentials form a good basis, the integrands themselves must be equals, so

$$\tilde{G}(\omega; k) = \frac{-1}{\omega^2 - k^2}. \tag{5.3.3}$$

Hence, if we can compute the inverse Fourier transform of $\tilde{G}$, we would have our Green's function. There is, however, as is usually the issue with this Green's function, there are a pair of poles on the real axis. So, we have to use the standard trick of introducing a positive quantity ϵ to push the poles off the real axis for the integration, and then take the limit $\epsilon \to 0$ when we are done. Essentially, we write

$$\tilde{G}(\omega; k) = \frac{-1}{(\omega + i(ik \pm \epsilon))(\omega - i(ik \pm \epsilon))}, \tag{5.3.4}$$

so the poles are at $\omega_R = k \pm i\epsilon$ and $\omega_L = -k \mp i\epsilon$ (labeled left and right). We note that there is an ambiguity in whether we add or subtract ϵ from ik, corresponding to which of the two poles is moved into the upper half-plane and which is moved into the lower half-plane.

The Fourier transform we now need to compute is

$$G^{\pm}(y; k) = \int \frac{\mathrm{d}\omega}{2\pi} \frac{-e^{i\omega y}}{(\omega - \omega_L)(\omega - \omega_R)}. \tag{5.3.5}$$

In order to evaluate this integral, we will need to use the Cauchy integral formula,

$$2\pi i f(a) = \oint_C \mathrm{d}z \frac{f(z)}{z - a} \tag{5.3.6}$$

[10] Strictly speaking, we should first Fourier transform x and x' separately and then deduce from the Fourier transform of the delta function that G depends only on the difference, but since this is typically the case, we simply assume this result.

where C is a simple, closed, counter-clockwise oriented curve whose interior contains a and no poles of $f(z)$. However, to use the Cauchy formula, we will need to transform our Fourier integral into an integral over a closed contour. What we can do is consider the semicircle of radius a which runs along the real axis and then into the upper or lower complex plane. If we suppose that $y > 0$, then $\Re[i\omega y] = -y\Re[\omega]$ is a negative quantity for ω in the upper half-plane. This means that as we let $a \to \infty$ on our contour, the contribution to the integral due to the arc will vanish and the value of the contour integral will be exactly the Fourier integral we are interested in. In exactly the same way, if $y < 0$ we need to close choose the semicircle to close in the lower half-plane, but we do need to be careful that here the contour is taken in the clockwise direction, which will result in an overall negative sign.

Let us first suppose that $y > 0$ so we are able to use the contour in the upper half-plane. To use the Cauchy integral formula, we then need to know which pole is located in the upper half-plane. If we take the negative sign for our additions of ϵ, there will be a pole at $\omega_L = -k + i\epsilon$, so we write

$$G^-(y > 0; k) = \frac{-1}{2\pi} \oint_C \frac{\frac{e^{i\omega y}}{(\omega - \omega_R)}}{\omega - \omega_L} = -i\frac{e^{-iky - y\epsilon}}{-2k} = -\frac{e^{-iky}}{2ik}, \tag{5.3.7}$$

where we have passed the limit $\epsilon \to 0$. In much the same way, we find

$$G^-(y < 0; k) = \frac{1}{2\pi} \oint_C \frac{\frac{e^{i\omega y}}{(\omega - \omega_L)}}{\omega - \omega_R} = i\frac{e^{iky + y\epsilon}}{2k} = -\frac{e^{iky}}{2ik}. \tag{5.3.8}$$

Hence,

$$G^-(|x - x'|; k) = -\frac{e^{-ik|x-x'|}}{2ik}. \tag{5.3.9}$$

If we were instead to take the plus sign on the additional factors of ϵ, we would have found

$$G^+(|x - x'|; k) = \frac{e^{ik|x-x'|}}{2ik}. \tag{5.3.10}$$

Now that we have the Green's function for the inhomogeneous Klein-Gordon equation, we are free to write the solution $\psi_k(x) = e^{ikx} + \int \mathrm{d}x' G^+(|x - x'|; k)J(x')$, which is a solution to the homogeneous problem plus the particular solution to the inhomogeneous problem given by the Green's function. We note that the most general solution to the problem at hand would involve both Green's functions and both plane waves, $e^{\pm ikx}$. However, we are only interested in scattering of an incoming wave. It is typical to interpret the positive sign in the complex exponential as indicating an "incoming" wave, which is why we have selected e^{ikx} for our homogeneous solution and G^+ as our Green's function.

This is where we come to the dirty trick alluded to earlier. Recall that the source term $J(x) = \int \mathrm{d}x'' u(x)u(x'')\psi_k(x'')$ is dependent on the wave function our Green's function was supposed to find for us. We may write

$$\psi_k(x) = e^{ikx} + \int \mathrm{d}x' G^+(x, x'; k)u(x') \int \mathrm{d}x'' u(x'')\psi_k(x''). \tag{5.3.11}$$

Let us now apply the operator $\int \mathrm{d}x u(x)$ to both sides of the above equation and liberally relabel dummy variables to find

$$\int \mathrm{d}x u(x)\psi_k(x) = \int \mathrm{d}x u(x)e^{ikx} + \left(\int \mathrm{d}x u(x)\psi_k(x)\right) \int \mathrm{d}x' \mathrm{d}x'' G^+(x', x''; k)u(x')u(x''). \tag{5.3.12}$$

The interesting thing about this particular equation, is that it allows us to solve for the quantity $a_k = \int \mathrm{d}x u(x)\psi_k(x)$ by writing

$$a_k = \frac{\tilde{u}^*(k)}{1 - K}, \qquad K = \int \mathrm{d}x' \mathrm{d}x'' G^+(x', x''; k)u(x')u(x''), \tag{5.3.13}$$

where $\tilde{u}^*(k) = \int \mathrm{d}x e^{ikx} u(x)$ is the conjugate of the Fourier transform of $u(x)$, using that $u(x)$ is a real-valued function. We note that K depends only on the Green's function and the function $u(x)$, not the wave function. So, we are now free to write the solution to the problem at hand as

$$\psi_k(x) = e^{ikx} + \frac{\tilde{u}^*(k)}{1 - K} \int \mathrm{d}x' G^+(|x - x'|; k)u(x'), \tag{5.3.14}$$

which is now a properly explicit solution: the solution is no longer implicitly defined by an integral equation.

Needless to say, this problem is unreasonable for allotted 45 minutes, and we are only half-way through it.

5.4 Part (c)

For this, we must remember that

$$r_k \equiv \lim_{x \to -\infty} \frac{\psi_k(x)}{e^{-ikx}}, \qquad t_k \equiv \lim_{x \to \infty} \frac{\psi_k(x)}{e^{ikx}}. \tag{5.4.1}$$

If we look to calculate the reflection coefficient first, we have[11]

$$r_k = \frac{\tilde{u}^*(k)}{1 - K} \int \mathrm{d}x' \frac{u(x')}{2ik} \lim_{x \to -\infty} \exp\left(ik(|x - x'| + x)\right). \tag{5.4.2}$$

[11] We note that all Fourier transforms are technically defined with $\lim_{\epsilon \to 0^+} e^{-ikx - \epsilon|x|}$ implicit, so in the limit $x \to \pm\infty$ of $e^{\pm ikx}$ vanishes.

But, $|x-x'| = \sqrt{(x-x')^2} = |x|\sqrt{1-2xx'/|x|^2+x'^2/x^2}$, so the expansion for large $|x|$ is then $|x-x'| \approx |x|\,(1-xx'/|x|^2) = |x| - \mathrm{sgn}(x)x'$. Thus,

$$\lim_{x\to-\infty} |x-x'| + x = \lim_{x\to-\infty} -x + x' + x = x'. \tag{5.4.3}$$

It now follows that the reflection coefficient is given by

$$r_k = \frac{1}{2ik}\frac{\tilde{u}^*(k)}{1-K}\int \mathrm{d}x' e^{ikx'}u(x') = \frac{\tilde{u}^*(k)}{2ik}\frac{\tilde{u}^*(k)}{1-K}. \tag{5.4.4}$$

The transmission coefficient follows by similar arguments, and is given by

$$t_k = 1 + \frac{1}{2ik}\frac{|\tilde{u}(k)|^2}{1-K}. \tag{5.4.5}$$

5.5 Part (d)

Lastly, we must check that $|r_k|^2 + |t_k|^2 = 1$. Expanding these terms, we find

$$|r_k|^2 = \frac{|\tilde{u}(k)|^4}{4k^2|1-K|^2}, \qquad |t_k|^2 = 1 + \frac{|\tilde{u}(k)|^2}{2ik}\frac{K-K^*}{|1-K|^2} + \frac{|\tilde{u}(k)|^4}{4k^2|1-K|^2}. \tag{5.5.1}$$

So, to satisfy the condition $|r_k|^2 + |t_k|^2 = 1$, it is both necessary and sufficient that

$$K - K^* = \frac{|\tilde{u}(k)|^2}{ik}. \tag{5.5.2}$$

So, we want to compute

$$K - K^* = \frac{1}{2ik}\int \mathrm{d}x\mathrm{d}x' u(x')\left[e^{ik|x-x'|} + e^{-ik|x-x'|}\right]u(x). \tag{5.5.3}$$

The absolute values in this expression make things look hopeless, but fortunately, we know the result is relatively simple. The trick of the matter is to notice that we are free to represent the absolute values in terms of theta functions by

$$e^{ik|x-x'|} = \theta(x-x')e^{ik(x-x')} + \theta(x'-x)e^{-ik(x-x')}. \tag{5.5.4}$$

With this, we may write the sum of exponentials in the integrand of (5.5.3) as

$$\begin{aligned} e^{ik|x-x'|} + e^{-ik|x-x'|} =& \theta(x-x')e^{ik(x-x')} + \theta(x'-x)e^{-ik(x-x')} \\ &+\theta(x-x')e^{-ik(x-x')} + \theta(x'-x)e^{ik(x-x')}, \end{aligned} \tag{5.5.5}$$

which we may now regroup terms to write

$$\begin{aligned} e^{ik|x-x'|} + e^{-ik|x-x'|} &= \left[\theta(x-x') + \theta(x'-x)\right] e^{ik(x-x')} \\ &+ \left[\theta(x-x') + \theta(x'-x)\right] e^{-ik(x-x')} \\ &= e^{ik(x-x')} + e^{-ik(x-x')}. \end{aligned} \tag{5.5.6}$$

Hence,

$$\begin{aligned} K - K^* &= \frac{1}{2ik} \int \mathrm{d}x\mathrm{d}x' u(x') \left[e^{ik(x-x')} + e^{-ik(x-x')}\right] u(x) \\ &= \frac{1}{2ik} \left[\tilde{u}(k)\tilde{u}^*(k) + \tilde{u}^*(k)\tilde{u}(k)\right] = \frac{|u(k)|^2}{ik}, \end{aligned} \tag{5.5.7}$$

exactly as required for $|r_k|^2 + |t_k|^2 = 1$, which completes the proof.

6 Problem 5: Quantum Mechanics

6.1 Problem Statement (Variational Method)

Confinement of a quark anti-quark pair with masses much larger than the typical confinement scale may be described by a non-relativistic Hamiltonian with an attractive potential which is linear in the distance between the quarks, and given by,

$$H = \frac{\mathbf{p}^2}{2\mu} + \frac{\hbar^2 a^3}{2\mu}|\mathbf{r}|, \tag{6.1.1}$$

where μ is the reduced mass, and $a > 0$ is a constant related to the confinement scale.

(a) Estimate the ground state energy by using the variational method and a family of trial wave functions depending on one parameter $\lambda > 0$,

$$\psi_\lambda(\mathbf{r}) = \begin{cases} (\lambda^2 - \mathbf{r}^2), & |\mathbf{r}| < \lambda. \\ 0, & |\mathbf{r}| > \lambda. \end{cases} \tag{6.1.2}$$

(b) How can the variational method be used to estimate the energy of the first excited state with zero orbital angular momentum as well? Please give a careful explanation, but there is no need to perform any calculations.

6.2 Part (a)

The variational method more or less goes as follows: suppose the ground state of a system is unknown to us, but we do have some parametrized model which we expect to approximate the ground state wave function well. This approximation is then a function of the parameters, so our guess may be written $\psi_\lambda(\mathbf{r})$, using the notation of the problem statement. Then since we have an explicit expression for $\psi_\lambda(\mathbf{r})$, we are free to compute directly the energy $E_\lambda = \langle\psi_\lambda|H|\psi_\lambda\rangle/\langle\psi_\lambda|\psi_\lambda\rangle$. The parameter value λ which causes ψ_λ to best approximate the ground state energy is therefore also the value of λ which minimizes E_λ, since the ground state has the minimum energy. There are some issues with this technique, but we will save a discussion of these issues for the solution to part (b), since it is more relevant there.

We will first compute the normalization,

$$\langle\psi_\lambda|\psi_\lambda\rangle = \int_0^\lambda 4\pi r^2 \mathrm{d}(\lambda^2 - r^2)^2 = \frac{32\pi}{105}\lambda^7. \tag{6.2.1}$$

The numerical prefactors in these variational method problems tend to get out of hand. For the sake of completeness, we will carry through the exact numerical prefactors here, but it is recommended on an exam that the reader simply define constant variables to encapsulate the prefactors, rather than risk making some numerical error in simplifying fractions.

Next, we need to compute $\langle\psi_\lambda|H|\psi_\lambda\rangle$. Since the Hamiltonian may be written

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{\hbar^2 a^3}{2\mu}|\mathbf{r}| \tag{6.2.2}$$

in the position basis, it follows from the identity $\nabla^2\mathbf{r}^2 = 6$ that

$$H\psi_\lambda = 6\frac{\hbar^2}{2\mu} + \frac{\hbar^2 a^3}{2\mu}|\mathbf{r}|(\lambda^2 - \mathbf{r}^2). \tag{6.2.3}$$

Hence,

$$\langle\psi_\lambda|H|\psi_\lambda\rangle = \frac{4\pi\hbar^2}{2\mu}\int_0^\lambda r^2 \mathrm{d}r(\lambda^2 - r^2)(6 + a^3 r(\lambda^2 - r^2)) = \frac{2\pi\hbar^2}{\mu}\left(\frac{4}{5}\lambda^5 + \frac{a^3}{24}\lambda^8\right). \tag{6.2.4}$$

From here it follows that the energy is given by

$$E_\lambda = \frac{7\hbar^2}{128\mu}\left(96\lambda^{-2} + 5a^3\lambda\right). \tag{6.2.5}$$

To minimize this, we take impose $\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = 0$, so

$$0 = \frac{7\hbar^2}{128\mu}\left(-192\lambda^{-3} + 5a^3\right), \tag{6.2.6}$$

which admits the real solution

$$\lambda = \frac{4}{a}\left(\frac{3}{5}\right)^{1/3}. \tag{6.2.7}$$

Our estimate of the ground state energy is now

$$E = \frac{7\sqrt[3]{75}}{64}\frac{(1+2a^2)\hbar^2}{\mu}. \tag{6.2.8}$$

6.3 Part (b)

In the beginning of part (a), we mentioned that there are some issues with the variational method. Essentially, if our guess for the approximate form turns out to be orthogonal to the true ground state, then the minimum energy we will compute will actually be a better approximation of the lowest excited state non-orthogonal to our guess – the variational method only guarantees that the energy we find will be bounded below by the true ground state energy.

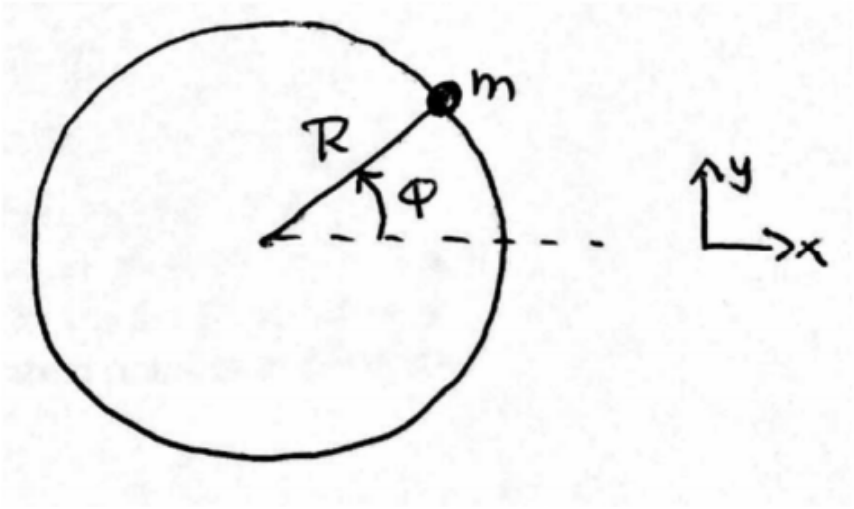
This may, however, be turned into an advantage for us if our goal is to compute excited state energies. For example, if we knew that the first excited state was parity odd while the ground state was parity even, then by choosing a trial function which is guaranteed to be parity odd, we know that the energy we find will be bounded below by the energy of the lowest non-orthogonal state, which we hope will be the lowest parity odd state, which would be the first excited state that we are interested in.

7 Problem 6: Quantum Mechanics

7.1 Problem Statement (Constrained Systems/Time Dependent Perturbation Theory)

A particle of mass m is constrained to move on a ring of radius R lying in the xy-plane. The system is in its ground state. A time-dependent potential is applied to the particle of the form

$$V(y,t) = \begin{cases} 0, & t < 0. \\ yV_0 e^{-t/\tau}, & t > 0. \end{cases} \tag{7.1.1}$$



At long times $t/\tau \gg 1$, the system is observed. Find the probability of finding it in each of the excited states of the unperturbed Hamiltonian.

You may assume that V_0 is sufficiently small that only first-order perturbation theory is necessary.

7.2 Solution

The first step of this problem is to determine the Hamiltonian of this constrained system. To find this, we will first write the Lagrangian in appropriate coordinates, from which we will compute the Hamiltonian. Once we have our Hamiltonian in hand, we will quantize the system, which we will find to be fairly simple since there is only two terms and all factors in the terms commute as operators. Taking the coordinate transformation $x = R\cos\phi, y = R\sin\phi$, we compute $\dot{x}^2 + \dot{y}^2 = R^2\dot{\phi}^2$, so the Lagrangian is

$$L = \frac{1}{2}mR^2\dot{\phi}^2 - \theta(t)V_0 R\sin\phi e^{-t/\tau}, \tag{7.2.1}$$

where $\theta(t)$ is the Heaviside theta. We immediately find that the conjugate momenta to the angular variable is $L_z = mR^2\dot{\phi}$. The Hamiltonian is therefore

$$H = L_z\dot{\phi} - L = \frac{L_z^2}{2mR^2} + \theta(t)V_0 R\sin\phi e^{-t/\tau}. \tag{7.2.2}$$

Alternatively, we could have just guessed that the kinetic term would be of the form $L_z^2/2I$ where $I = mR^2$ is the moment of intertia. It is convenient that this Hamiltonian factors

cleanly into two parts, $H = H_0 + V$. As promised, the two terms are composed only of factors that commute, so there are no issues just replacing ϕ and L_z by operators $\hat{\phi}$ and $\hat{L}_z$.

Before launching into the perturbation calculation, we review how the perturbation formula is derives, which will hopefully give us a way to help recall the formula itself. We begin by defining the operator $U(t, t_0)$ to be the operator which evolves a state in the Schrödinger picture from a time t_0 to a time t. With this, we are free to define the operator $U_I(t, t_0) = e^{iH_0 t/\hbar} U(t, t_0) e^{-iH_0 t_0/\hbar}$. We note that this is the operator U evolved *backwards* in time by the operator H_0. Furthermore, we will let $\{|n\rangle\}$ be the eigenbasis of the operator H_0 such that $H_0|n\rangle = E_n|n\rangle$.

We note that the transition amplitude may be written

$$\langle n|U(t, t_0)|i\rangle = \langle n|e^{-iH_0 t/\hbar} U_I(t, t_0) e^{iH_0 t_0/\hbar}|i\rangle = e^{-i(E_n t - E_i t_0)/\hbar}\langle n|U_I(t, t_0)|i\rangle. \tag{7.2.3}$$

It therefore follows that

$$|\langle n|U(t, t_0)|i\rangle|^2 = |\langle n|U_I(t, t_0)|i\rangle|^2, \tag{7.2.4}$$

so the transition probability may be calculated as a matrix element of U in the H_0 eigenbasis just as well as the matrix element of U_I. However, it is important to note that this only works when we are asking for the probability of transition between eigenstates of H_0. If we interested in the transition probability between two arbitrary states, we would need to be careful to track the energy phases that appear in (7.2.3).

This is all well and good, but it is only useful if we have some means of calculating the operator U_I. Since the operator U is necessarily generated by the full Hamiltonian H, it follows that $i\hbar\frac{\partial}{\partial t}U(t, t_0) = HU(t, t_0) = (H_0 + V)U(t, t_0)$. Therefore,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}U_I(t, t_0) &= -H_0 U_I + e^{iH_0 t/\hbar}\frac{\partial U}{\partial t}e^{-iH_0 t_0/\hbar} \\ &= -H_0 U_I + H_0 U_I + e^{iH_0 t/\hbar} V(t) U(t, t_0) e^{-iH_0 t\hbar} \\ &= e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar} U_I(t, t_0) \\ &\equiv V_I(t) U_I(t, t_0). \end{aligned} \tag{7.2.5}$$

This differential equation along with the initial condition $U_I(t, t_0)|_{t=t_0} = 1$ allows us to integrate the equation to find

$$\begin{aligned} U_I(t, t_0) &= 1 + \frac{-i}{\hbar}\int_{t_0}^{t} \mathrm{d}t' V_I(t') U_I(t, t_0) \\ &= 1 + \frac{-i}{\hbar}\int_{t_0}^{t} \mathrm{d}t' V_I(t') + \left(\frac{-i}{\hbar^2}\right)^2 \int_{t_0}^{t}\mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V_I(t') V_I(t'') + \cdots, \end{aligned} \tag{7.2.6}$$

where in the final line we have iterated on the integro-equation to form the Dyson series solution. However, it would be extremely rare for more than the first order correction to be necessary on this exam. Indeed, we are told in the problem statement above to only use the first order correction. If we were to take the matrix element of this solution, keeping only to first order then,

$$\langle m|U_I(t,t_0)|n\rangle = \delta_{mn} + \frac{-i}{\hbar}\int_{t_0}^{t} \mathrm{d}t' \langle m|V(t')|n\rangle e^{-i(E_n-E_m)t'/\hbar}. \tag{7.2.7}$$

We can remember this formula by remembering that the prefactor comes from iterating on a formula with $\frac{1}{i\hbar}$, and we can remember the order of the energies in the exponential by noticing that they are actually the same as those time-independent perturbation theory – The state of interest (in this case, meaning the state on the RHS) comes first. In fact, we can easily obtain the first order time-independent state correction from the time-dependent perturbation theory by inserting a complete basis of $|m\rangle$ states between V and the left hand exponential factor and performing the integral – pull out the assumed time independent matrix element from the integral, and perform the integral directly.

So, back to the problem at hand, we see that we will need the energies and eigenstates of the operator $H_0 = \frac{\hat{L}_z^2}{2I}$. If we suppose that $|\psi\rangle$ is an angle basis state, then we are free to make use of the representation $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$. The eigenvalue equation for the angle-basis wave function is therefore

$$\frac{\partial^2}{\partial\phi^2}\psi(\phi) = -\frac{2IE}{\hbar^2}\psi(\phi), \tag{7.2.8}$$

which admits solutions[12]

$$\psi(\phi) = \frac{1}{\sqrt{2\pi}}e^{-i\omega\phi}, \qquad \omega^2 = \frac{2IE}{\hbar^2}. \tag{7.2.9}$$

However, we do have a boundary condition to impose, which is $\psi(2\pi) = \psi(0)$, which then requires $2\pi\omega = 2\pi m$. Hence, we find that $\omega \in \mathbb{Z}$. It now makes sense to label the states by the particular integer m, so we have frequencies $\omega_m = m$ which imply states $\langle\phi|m\rangle = \psi_m(\phi) = e^{-im\phi}/\sqrt{2\pi}$ and energies $E_m = \frac{m^2\hbar}{2I}$. The ground state of this system is therefore $E_0 = 0$.

The problem asks us to compute the probability of transitioning from the ground state to any excited state with the approximation $t = \infty$. By our above general discussion, this

[12]The factor of $1/\sqrt{2\pi}$ is present only for normalization. Alternatively, we could have defined the measure on the space of ϕ to be $\mathrm{d}\phi/2\pi$, in which case we would not have the $1/\sqrt{2\pi}$ in the definition of the wave function.

means we need to compute the modulus squared of $\langle m|U(\infty,0)|0\rangle$. For this, we first need to compute the matrix element $\langle m|V|0\rangle$, which we may write in the $|\phi\rangle$ basis as

$$\begin{aligned}\langle m|V|0\rangle &= \theta(t)V_0Re^{-t/\tau}\int_0^{2\pi}\mathrm{d}\phi\psi_m(\psi)\sin\phi\psi_0(\phi)\\ &= \theta(t)V_0Re^{-t/\tau}\int_0^{2\pi}\frac{\mathrm{d}\phi}{4\pi i}e^{-im\phi}\left(e^{i\phi}-e^{-i\phi}\right)\\ &= \theta(t)V_0Re^{-t/\tau}\frac{1}{2i}\left(\delta_{m,1}-\delta_{m,-1}\right).\end{aligned} \tag{7.2.10}$$

We now compute the integral (7.2.7) for $n=0$ for $m\neq 0$ so we may ignore the Kronecker delta[13]. We find

$$\begin{aligned}\langle m|U_I(\infty,0)|0\rangle &= -\frac{V_0R}{2\hbar}(\delta_{m,1}-\delta_{m,-1})\int_0^\infty \mathrm{d}t'\exp\left[-\left(\frac{1}{\tau}+i\omega_{0m}\right)t'\right]\\ &= \frac{V_0R}{2\hbar}(\delta_{m,1}-\delta_{m,-1})\frac{\tau}{1+i\omega_{0m}\tau},\end{aligned} \tag{7.2.11}$$

where we have define $\omega_{nm}=(E_n-E_m)/\hbar$ for brevity. Since we know $m=\pm1$, however, we are free to write $\omega_{0,\pm1}=-\frac{\hbar}{2I}$. This now allows us to write

$$\langle\pm1|U_I(\infty,0)|0\rangle = \frac{V_0R}{2\hbar}(\delta_{m,1}-\delta_{m,-1})\frac{2I\tau}{2I-i\tau\hbar}. \tag{7.2.12}$$

Thus we find,

$$P(0\to\pm1) = \left(\frac{V_0R}{\hbar}\right)^2\frac{I^2\tau^2}{4I^2+\tau^2\hbar^2} \tag{7.2.13}$$

for the transition probabilities. All other transition probabilities are zero to within order V_0^2.

[13]Note that it is not meaningful to try checking that probabilities sum to unity since we are computing these amplitudes in perturbation theory, which is not constructed to be unitary at every orders. I also conjecture that that these "transition probabilities" are likely better interpreted as integrated transition rates. If we adopt this view, then it is immediately clear from the theory of continuous time Markov chains that the transition rates integrated to finite order should not be expected to conserve probability unless we are dealing with an exceptionally boring process. This is, however, outside the generally accepted cannon, the knowledge of which this exam is supposed to test.

8 Problem 7: Quantum Mechanics

8.1 Problem Statement (Quantization of Constrained Systems/Commutator Algebras)

Consider a particle of charge e and mass m_0 which is constrained to move on the surface of a sphere of radius R (we do not consider spin in this problem). There is a uniform magnetic field $\mathbf{B}$.

(a) Write the Hamiltonian in terms of the momentum and angular momentum operators, neglecting terms of second order in the field.

(b) Find the energy levels of the system.

(Hint: work in the gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$).

8.2 Part (a)

Though the easiest way to deal with constrained system is to do as we did in the previous problem and start from an appropriate classical Lagrangian, this is not always a feasible strategy on a timed exam. For this problem, it will be far simpler to start with the Hamiltonian for a free particle coupled to a magnetic field, and look to convert it directly to spherical coordinates with the constraint $r = R$ is constant[14].

The Hamiltonian for the unconstrained system would be

$$H = \frac{1}{2m_0}|\mathbf{p} - e\mathbf{A}|^2 \approx \frac{1}{2m_0}\mathbf{p}^2 - \frac{e}{2m_0}(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}), \tag{8.2.1}$$

where we have neglected the term which is second order in the vector potential. Now, we are free to write $\mathbf{p}^2 = p_r^2 + \frac{1}{r^2}\mathbf{L}^2$ for the momentum in spherical coordinates. But, as mentioned above, we take $r = R$ to be constant, so $p_r = 0$ and the momentum is just $\mathbf{p}^2 = \frac{1}{R^2}\mathbf{L}^2$.

Next, choosing the suggested gauge, we are free to write

$$p_n A^n = \frac{1}{2}p_n\epsilon^{nmk}B_m x_k = \frac{1}{2}B_m\epsilon^{mkn}(x_k p_n - i\hbar\delta_{kn}) = \frac{1}{2}\mathbf{B}\cdot(\mathbf{r}\times\mathbf{p}) = \frac{1}{2}\mathbf{B}\cdot\mathbf{L}. \tag{8.2.2}$$

By a nearly identical argument, we also find $\mathbf{A}\cdot\mathbf{p} = \frac{1}{2}\mathbf{B}\cdot\mathbf{L}$. It now follows that the Hamiltonian may be written

$$H = \frac{1}{2m_0R^2}\mathbf{L}^2 - \frac{e}{2m_0}\mathbf{B}\cdot\mathbf{L}. \tag{8.2.3}$$

[14] I do not believe this strategy works in general. The general theory for quantization of constrained systems was worked out by Dirac and, though I am not familiar with all the details thereof, I am aware that it is a touch more complicated in general.

8.3 Part (b)

Now that we have a Hamiltonian written in terms of the angular momentum operator, we are free to choose, without loss of generality, the magnetic field to point in the z-direction so the Hamiltonian becomes

$$H = \frac{1}{2m_0R^2}\mathbf{L}^2 - \frac{eB_z}{2m_0}L_z. \tag{8.3.1}$$

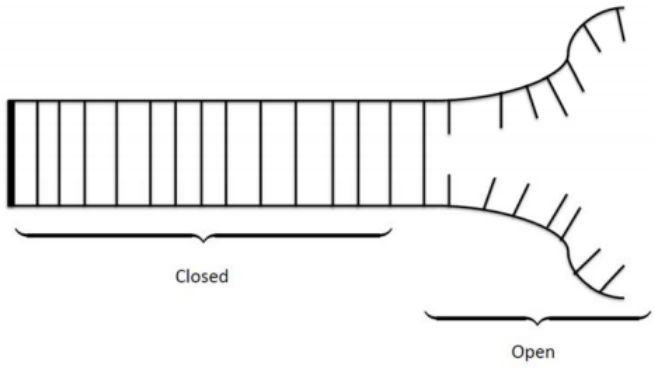
It should be clear at this point that the eigenvectors of our Hamiltonian are the angular momentum vectors, $|\ell, m\rangle$ with $\mathbf{L}^2|\ell, m\rangle = \hbar^2\ell(\ell+1)|\ell, m\rangle$ and $L_z|\ell, m\rangle = m\hbar|\ell, m\rangle$. Thus, the energies are also labeled by ℓ and m, and may be written

$$E_{\ell m} = \frac{\ell(\ell+1)\hbar^2}{2m_0R^2} - \frac{em\hbar B_z}{2m_0}. \tag{8.3.2}$$

9 Problem 8: Statistical Mechanics

9.1 Problem Statement (Molecular Zipper)

A molecular zipper. Consider the following simple model for the melting of DNA. Two unbreakable molecular strands are coupled by N links. See the figure below. Each link can be in one of $G+1$ states. Of these states, G are unbound (open) and have energy ϵ, and one is bound (closed) and has energy 0. You must assume that links unbind sequentially. In other words, the only states of the system with finite energy are those in which all links to the right of a given link are unbound. Let $n \leq N-1$ be the number of unbound links.



(a) Find the partition function of the system.

(b) Find the mean fraction of unbound links $\chi = \frac{\langle n \rangle}{N}$ at temperature T in the thermodynamic limit $N \to \infty$. Show that there is a critical temperature where this mean fraction changes in the thermodynamic limit.

(c) Compute the fraction of completely unzipped ($n = N - 1$) zippers in a noninteracting solution of these molecules as a function of temperature T.

9.2 Part (a)

For this problem, it is not possible to use the usual trick for non-interacting systems of computing the single-particle partition function and then raising it to a power. If we were to attempt this trick here, we would find that it does not prohibit states with non-sequential zippering.

Instead, we note that the energy of the system is uniquely identified by the number of unbound links[15]. If the number of open links is n, then the energy of the system is necessarily $n\epsilon$. If we know that n links are open, then each link may be in one of G states. We do not place any restriction on which of the G states each of the links must be in once we know that the link is open, so each of the links may have its particular state chosen independently from the G possibilities. This means that there are G^n possible specific states for the system to be in when n links are open.

Since the partition function is simply the sum over states, it follows that[16]

$$Z = \sum_{n=0}^{N-1} G^n e^{-\beta n\epsilon} = \sum_{n=0}^{N-1} x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=N}^{\infty} x^n = \frac{1 - x^N}{1 - x}, \tag{9.2.1}$$

where $x \equiv Ge^{-\beta\epsilon}$ and we have rederived the formula for the finite geometric sum from the infinite geometric sum because I can never remember the finite version.

[15] A good general strategy for dealing with with sample space restrictions like in this problem is to first find a way of uniquely indexing the energies, then counting the degeneracy of a given energy.

[16] A key step in all physics problems of these exams is to ignore questions of convergence. Indeed, if we look carefully, we will see that the critical temperature in part (b) is the temperature at which the geometric sum no longer converges for $N \to \infty$.

9.3 Part (b)

For this part, we just need to compute directly,

$$\langle n \rangle = \sum_{n=0}^{N-1} n \frac{x^n}{Z} = \frac{\partial}{\partial x} \ln Z = \frac{x}{1-x} - \frac{(N+1)x^N}{1-x^N}, \tag{9.3.1}$$

for arbitrary N. For the large N limit, we write

$$\lim_{N\to\infty} \frac{\langle n \rangle}{N} = \lim_{N\to\infty} \frac{1}{1-x^{-N}} = \begin{cases} 1, & x > 1. \\ 0, & x < 1. \end{cases} \tag{9.3.2}$$

That is, in the $N \to \infty$ limit, the average fraction converges to either 100% or 0% linked. The condition for the critical condition is therefore $x = 1$, which implies

$$T_c = \frac{k\epsilon}{\ln G} \tag{9.3.3}$$

is the critical temperature for the transition.

9.4 Part (c)

By the law of the unconscious statistician, we expect that if we draw a large number of samples from our distribution, the fraction drawn with probability P will converge to the value of that probability[17]. Therefore, what we are asked to compute is actually,

$$P(n = N-1) = \frac{x^{N-1}}{Z} = \frac{1-x}{1-x^N} x^{N-1}. \tag{9.4.1}$$

And that's it.

10 Problem 9: Statistical Mechanics

10.1 Problem Statement (Stephan-Boltzmann Law)

You have 3 concentric spheres with radii $R_1 < R_2 < R_3$. The sphere at R_1 is maintained at temperature T_1 and the sphere at R_3 is maintained at temperature T_3. Assume that the spheres are black and that the only heat transport occurs via photons.

[17] We should always remember that nearly all physicists are frequentists.

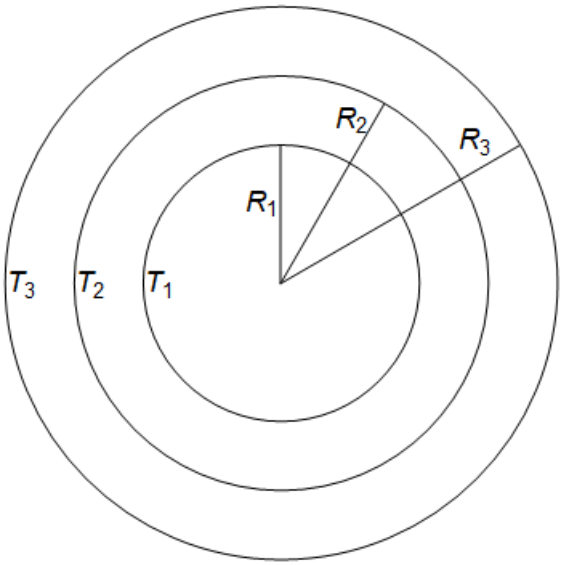
(a) Draw a diagram (or two) and label it with the variables you will use to solve the problem.

(b) Find the stead-state temperature T_2 of the sphere at R_2.

(c) Evaluate the interesting limiting cases of your result from (b).

10.2 Parts (a)-(c)

So first of all, we will point out that the color of the spheres being black tells us nothing about the problem. We should, however, interpret this to mean we are supposed to treat the spheres as black-body radiators. Since heat transport only occurs via this radiation, we know that we will need to use the Stephan-Boltzmann law to determine the radiated powers. First, however, we are supposed to draw a diagram. Strictly speaking, the correct way to draw a diagram which includes only that which will be useful for solving the problem is to solve the problem and then draw a diagram labeled with those things which proved to be useful. However, we shouldn't complain too much, because if we didn't know how to actually solve the problem, we might hope to get some partial credit by guessing the perhaps-obvious things, the temperatures, radii, and maybe the surface areas.



(10.2.1)

So, here's the idea. If the entire system is in a steady-state configuration, then the middle sphere must be radiating power equal to the power it is absorbing. We know that the power

radiated by the center sphere is given by the Stephan-Boltzmann law, $P_2 = \sigma(2A_2)T_2^4$ for some material-dependent σ, the area being doubled because we are radiating from both sides of the surface.

The other two surfaces radiate the same, but we have to be careful that we only count the radiated power absorbed by the middle sphere. For this power radiated by the inner sphere, everything radiated on the R_2 side is also absorbed by the middle sphere. So, the power absorbed from the inner-most sphere is just $P_1 = \sigma A_1 T_1^4$. When we look at the outer sphere, however, some of the radiated power can miss. That is, if we imagine the power radiated from a particular point on the outer sphere, the power is radiated evenly across all solid angles, and the cross-section of the middle sphere only occupies a portion of the full solid angle. At the end of the day, due to the spherical symmetry of the system, the power absorbed from the outer sphere is $P_3 = \sigma A_2 T_3^4$.

The same issue that happens for the power radiated from the outer to the middle happens when radiating from the middle to the inner. Since all power from the middle not absorbed by the inner is reabsorbed by the middle, the actual power emitted by the middle and *not reabsorbed* is $P_2 = \sigma(A_2 + A_1)T_2^4$. Since $P_2 = P_1 + P_3$, it follows that

$$T_2^4 = \frac{R_1^2 T_1^4 + R_2^2 T_3^4}{R_1^2 + R_2^2}. \tag{10.2.2}$$

For the final part we are supposed to evaluate "interesting" limiting cases. Unfortunately, I do not know what the author of this question, nor the grader, might find interesting, so we will instead just look at some random limiting cases. For example, we could take $R_1 = R_2$ and find that $T_2^4 = \frac{1}{2}(T_1^4 + T_3^4)$, so the temperature to the fourth is just the average of the two temperatures to the fourth. If $R_1 = 0$, then there isn't an inner sphere, and the temperature of the middle sphere is just equal to the temperature of the background radiation, which since the formula we found does not depend on R_3, might as well be the temperature at infinity.

11 Problem 10: Statistical Mechanics

11.1 Problem Statement (Bose-Einstein Condensation)

(a) Find the temperature dependence of the heat capacity at constant volume (C_V) for an ideal Bose gas below the BE condensation temperature.

(b) Give an expression for the condensation temperature T_0 in terms of N, V. Definite integrals which are just numerical factors may be left indicated as such.

11.2 Part (a)

Before launching into the calculation, we first review the relevant points of Bose-Einstein condensation. The simple most important quantity at our disposal is the occupation number,

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + a}, \tag{11.2.1}$$

where $a = -1$ for Bosons, $a = +1$ for Fermions, and $a = 0$ for classical Maxwell-Boltzmann systems. Though we only require the Bose case for this problem, it is helpful to remember all three together. This is because we can remember that Bosons get a minus sign because they correspond to commutators, which have a minus sign, while Fermions get a plus sign because they correspond to anticommutators, which have a plus sign. The classical Maxwell-Boltzmann particles correspond to neither, so they get a zero. The Maxwell-Boltzmann case then further fixes the sign in the exponential, since we know $n_{MB}(\epsilon) \propto e^{-\beta(\epsilon-\mu)}$, which is just the Boltzmann factor. So, keeping all three together in our minds in this way, all relevant signs are fixed uniquely by surrounding knowledge.

Next, we know that the sum of the occupation numbers must be the expected number of particles, so

$$N = \sum_{\epsilon} n(\epsilon) = \sum_{\epsilon} \frac{1}{e^{\beta(\epsilon-\mu)} - 1}, \tag{11.2.2}$$

where the summation is taken over the collection of discrete energy states of the system. For large systems, however, we would like to approximate this summation by an integral. The correct way to make this approximation is rather involved, so instead we will just point out that in the case $\mu = 0$, the $\epsilon = 0$ term in the above summation goes to infinity. This is the source of Bose-Einstein condensation, but for the moment is just something our integral approximation will not handle very well, so we will just agree to pull out the $\epsilon = 0$ term ahead of time and write[18]

$$N = \frac{1}{z^{-1} - 1} + \sum_{\epsilon \neq 0} \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \approx \frac{z}{1-z} + \int \frac{\mathrm{d}^3x \mathrm{d}^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\epsilon(p)-\mu)} - 1} \tag{11.2.3}$$

where $z \equiv e^{\beta\mu}$. It is fairly standard to instead state the integral above as an integral over the energies by using $\epsilon = \frac{1}{2m}p^2$ to change variables, However, though it may be standard, it is

[18] In this expression and for the remainder of the problem, we write the energy as a function of only the momenta. Generally speaking, it is free to be a function of both the position and momenta, but this is almost never the case in examples of interest. No steps in this general discussion assume the energy is position independent, we only even keep the momenta dependence to remind the reader that this is a function we are integrating over, not a parameter which N depends on, as is the case for the occupation numbers, $n(\epsilon)$.

almost never a useful form for the integral, and indeed as soon as we attempt to modify the system from the free-particle case in any way, becomes false. Furthermore, I think (11.2.3) is easier to remember since the measure is exactly the same as the measure for the partition function. If necessary, the appropriate change of variables may be made quickly enough.

It is also standard to write $N_0 \equiv \frac{z}{1-z}$, so we have

$$N - N_0 = \int \frac{\mathrm{d}^3x\mathrm{d}^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\epsilon(p)-\mu)} - 1} \equiv N_e \tag{11.2.4}$$

to express the number of particles which are not condensed.

Now that we have an expression for the number of particles in excited states for a Bose-Einstein gas, we are in a position to discuss condensation. The typical language used to describe condensation makes it sound as though, at some critical temperature, the system undergoes some kind of a phase transition and suddenly $N_e \to 0$ while $N_0 \to \infty$. While the process of condensation could be viewed as a type of phase transition, the rest of the statement just given is nonsense. There is no discrete or sudden change.

Instead, the idea of Bose-Einstein condensation properly stems from the observation that N_e, when viewed as a function of $z = e^{\beta\mu}$ is a monotonically increasing function which takes a maxima at $z = 1$ (which is the maximum allowable value for z in a Bose-Einstein system anyway). More than being monotonically increasing, it is bounded[19]. To show this, we need only compute

$$\frac{\partial N_e}{\partial z} = \int \frac{\mathrm{d}^3x\mathrm{d}^3p}{(2\pi\hbar)^3} \frac{e^{\beta\epsilon(p)}}{(e^{\beta\epsilon(p)} - z)^2}. \tag{11.2.5}$$

The integrand is manifestly positive for all $0 \leq z \leq 1$, so we know the integral, and hence $\frac{\partial N_e}{\partial z} \geq 0$ for all z of interest. Indeed, since the integrand is also never zero, the inequality is strict, so $\frac{\partial N_e}{\partial z} > 0$. Thus, the function $N_e(z)$ is strictly monotonically increasing on $0 \leq z \leq 1$, and as mentioned before, we assume $N_e(1)$ to exist.

Now that we know N_e is bounded above by some number, we are free to ask what happens if $N > \max\{N_e(z)|z \in [0,1]\}$. By the pigeon-hole principle, it would follow that $N_0 \neq 0$, so there must be a non-negligible number of particles exactly in the ground state. This is what we define to be Bose-Einstein condensation. If we accept this definition, then the condition

$$N = \int \frac{\mathrm{d}^3x\mathrm{d}^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta\epsilon(p)} - 1} = N_e(1) \tag{11.2.6}$$

defines the critical number of particles at which the maximum of N_e is attainable. Clearly, if the number of particles are increased beyond this number, we are guaranteed that some will be in exactly the ground state.

[19]This last point is really a matter of assuming that $\epsilon(p)$ is such that $N_e(z = 1)$ converges to a finite value, as is the case for the free particle case.

Another, and more common, way of looking at the condition (11.2.6) is that we could control N and increase the number of particles till we meet the critical number beyond which particles are forced into the ground state, or we could notice that the critical number $N_e(1)$ depends on the temperature and fix the number of particles N in order to determine a critical temperature. This is what is usually done.

Now, there is a fine point which we must point out now. Generally speaking $\mu = \mu(T)$ – the chemical potential is not an independent parameter of the system which we are allowed to tune. In particular, the above arguments imply that if T_c is the critical temperature satisfying (11.2.6), then $\mu(T_c) = 0$. The situation seems to be fairly complicated, and I am not, at the time of writing, aware of general arguments about the temperature behavior of μ and such. However, for the sake of this document, we will satisfy ourselves with the knowledge that, both empirically and in the free particle case, z is a monotonically decreasing function of the temperature. This means that for $T < T_c$, we have $z \approx 1$, and so in the same region, $\beta\mu(T) \approx 0$, which implies that, at low temperatures, $\mu(0) = \frac{\partial \mu}{\partial T}|_0 = 0$, so $\mu \approx 0$ to order $\mathcal{O}(T/T_c)^2$. Since we know $\mu(T_c) = 0$, it would be nice to expand about $T = T_c$, but the apparent approximate discontinuity in the first derivative of z makes it seem unlikely that such an expansion would do us much good.

If we accept these arguments, then it follows that condensation occurs for temperatures $T < T_c$. For temperatures $T > T_c$, we note that the arguments above only state that the critical condition implies condensation occurs beyond the critical point, which we argued means a temperatures below T_c. The above arguments do not, however, imply that condensation cannot occur at higher temperatures as well. Indeed, unless $z = 0$ exactly for $T > T_c$, $N_0 \neq 0$, which implies that there is some condensation. However, even for $z = 0.99$, we still only have $N_0 \approx 100$. One hundred particles is a fairly small number for most purposes, so we approximate N_0 as being negligible until condensation, so for $T > T_c$, $N \approx N_e$. Now, obviously, these are weak arguments which it likely would not be too difficult to find counter examples to. However, they are roughly true for the free particle, which is the only case anyone ever talks about. So, feel free to treat this last paragraph as a description to common arguments which need not be generally true, but are good enough for the most common example.

The description given above should be good enough to actually solve the problem now. Since we are assumed to be in the regime $T < T_c$, we are free to take $\mu = 0$ and $z = 1$. Since the particles in the ground state have zero energy, we are free to exclude these particles from our energy calculation. Hence,

$$\langle E \rangle = \int \frac{\mathrm{d}^3 x \mathrm{d}^3 p}{(2\pi\hbar)^3} \frac{\epsilon(p)}{e^{\beta\epsilon(p)} - 1}. \tag{11.2.7}$$

Since we are dealing with an ideal Bose gas, we have $\epsilon(p) = \frac{p^2}{2m}$. The integrand does not depend on the position, so we can immediately compute the spatial integrals, and are just left with the momenta integrals. But the integrand also only depends on the magnitude of the momenta, not the angle, so we convert to spherical coordinates, and immediately compute the integral over the solid angle to find

$$\langle E \rangle = \frac{4\pi V}{(2\pi\hbar)^3} \int_0^\infty p^2 \mathrm{d}p \frac{\epsilon(p)}{e^{\beta\epsilon(p)} - 1}. \tag{11.2.8}$$

But now $\mathrm{d}\epsilon = \frac{p}{m}\mathrm{d}p$ and $p = (2m\epsilon)^{1/2}$, so $\mathrm{d}\epsilon = \sqrt{\frac{2\epsilon}{m}}\mathrm{d}p$. Therefore,

$$\langle E \rangle = \frac{4\pi V}{(2\pi\hbar)^3}\sqrt{2m^3} \int_0^\infty \mathrm{d}\epsilon \frac{\epsilon^{3/2}}{e^{\beta\epsilon} - 1} = \frac{4\pi V}{(2\pi\hbar)^3}\frac{\sqrt{2m^3}}{\beta^{5/2}} \int_0^\infty \mathrm{d}x \frac{x^{3/2}}{e^x - 1}, \tag{11.2.9}$$

where now $\int_0^\infty \mathrm{d}x \frac{x^{3/2}}{e^x - 1} \equiv A_3$ is some constant whose value, $A_3 = \frac{3}{4}\sqrt{\pi}\zeta(5/2)$, is completely irrelevant.

Now that we have the expected energy, we may compute the heat capacity by $C_V = \frac{\partial\langle E\rangle}{\partial T}|_V$. Therefore,

$$C_V = \frac{4\pi V}{(2\pi\hbar)^3}\sqrt{2m^3}A_3 k^{5/2}\frac{\partial T^{5/2}}{\partial T} = \frac{10\pi V}{(2\pi\hbar)^3}\sqrt{2m^3}A_3 k^{5/2}T^{3/2} \propto T^{3/2}. \tag{11.2.10}$$

11.3 Part (b)

We now need to impose the condition (11.2.6) discussed earlier, and solve for the temperature T_c. We again change variables as before, and find

$$N = \frac{4\pi V}{(2\pi\hbar)^3}\sqrt{2m^3} \int_0^\infty \mathrm{d}\epsilon \frac{\epsilon^{1/2}}{e^{\beta_c\epsilon} - 1} = \frac{4\pi V}{(2\pi\hbar)^3}\frac{\sqrt{2m^3}}{\beta_c^{3/2}}A_1, \tag{11.3.1}$$

where $A_1 = \int_0^\infty \mathrm{d}\epsilon \frac{x^{1/2}}{e^x - 1}$ is again some constant whose value, $A_1 = \frac{\sqrt{\pi}}{2}\zeta(3/2)$, is completely irrelevant. It follows that

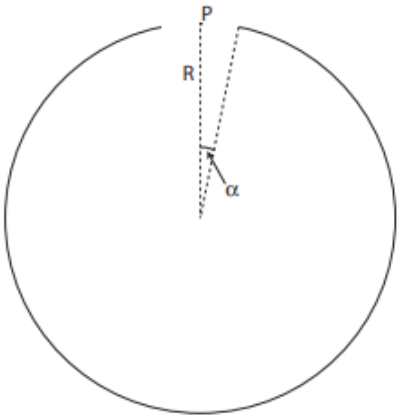
$$T_c = \left[\frac{N}{k^{3/2}}\frac{(2\pi\hbar)^3}{4\pi V}\frac{1}{\sqrt{2m^3}A_1}\right]^{2/3}, \tag{11.3.2}$$

which can almost certainly be made neater, but who cares.

12 Problem 11: Electromagnetism

12.1 Problem Statement (Electrostatic Calculation)

A thin spherical shell of radius R is uniformly charged so that the charge per unit area on the surface is σ. You take a sword and chop off the very top of the sphere, so that there is a hole in the apex with polar opening angle α, as shown below.



(a) If the angle α is small (so that the diameter of the opening is much smaller than the radius of the sphere), what is the electric field at the center of the sphere (magnitude and direction)?

(b) What is the electric field at point P in the diagram (in the opening, at the location where the apex of the sphere used to be before I sliced off the top)?

12.2 Part (a)

This exact problem also appeared on as problem 7 on the 2018 exam. Since I have already written a detailed solution for this problem in the 2018 entry, this entry will instead be dedicated to using a quicker approximation.

In the 2018 solution, we approximated the field as a completed sphere, which contributes no field in the interior region, plus a disk whose charge density negates that of the completed sphere at the hole. There, the approximation was to treat the curved section of surface as a flat disk. Here, however, since the hole is so small, we could also approximate that disk as a point charge of the same total charge. The total charge needed in the hole is approximately $q = -\pi\sigma(\alpha R)^2$.

The electric field of such a point particle when evaluated at the center of the sphere is then

$$\mathbf{E} = -\frac{\pi\sigma(\alpha R)^2}{4\pi\epsilon_0 R^2}(-\hat{\mathbf{z}}) = \frac{\sigma}{4\epsilon_0}\alpha^2\hat{\mathbf{z}}, \tag{12.2.1}$$

which is the same result we found from the disk solution by only keeping to lowest order in α.

12.3 Part (b)

The field at the point P can be imagined as the sum of the field due to a completed sphere plus the field due to an infinite plane with charge surface density $-\sigma$. Again, we know that the field inside the sphere is zero, so on that side of the imaginary situation, we have

$$\mathbf{E} = \frac{-\sigma}{2\epsilon_0}(-\hat{\mathbf{z}}) = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}. \tag{12.3.1}$$

In case we were worried that we might have made a mistake somewhere along the way, we can check to make sure our result is continuous at P, which we know it should be since the charge density there is supposed to be zero. So, if we considered the field just on the outside of the imaginary sphere, we would have

$$\mathbf{E} = \frac{-\sigma}{2\epsilon_0}\hat{\mathbf{z}} + \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}, \tag{12.3.2}$$

which matches what we found before and what was found in the 2018 solution.

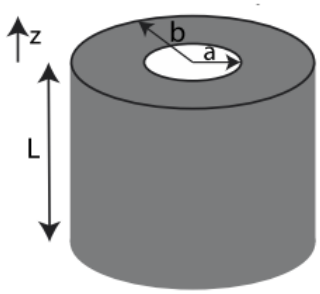
13 Problem 12: Electromagnetism

13.1 Problem Statement (Magnetostatic Calculation in Matter)

A permanent magnet is created from a hollow cylindrical piece of magnetic material with inner radius a, outer radius b, and length L. It is given magnetization

$$\mathbf{M} = \frac{M_0\rho}{a}\hat{\boldsymbol{\phi}}, \tag{13.1.1}$$

where ρ is the distance from the axis of the hollow cylinder and $\hat{\boldsymbol{\phi}}$ is the azimuthal unit vector.



(a) Where are the bound currents in this magnet and what are their values?

(b) What is the magnetic field produced by this magnet?

Useful formulas:

$$\begin{aligned} \nabla\psi &= \mathbf{e}_1\frac{\partial\psi}{\partial\rho} + \mathbf{e}_2\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3\frac{\partial\psi}{\partial z} \\ \nabla\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\rho A_1}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z} \end{aligned} \tag{13.1.2}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial\rho A_2}{\partial\rho} - \frac{\partial A_1}{\partial\phi}\right) \tag{13.1.3}$$

13.2 Part (a)

For this, we just need to remember that $\mathbf{J}_b = \nabla\times\mathbf{M}$ and $\mathbf{K} = \mathbf{M}\times\hat{\mathbf{n}}$ for the bound bulk and surface current densities. So, the bulk density is just

$$\mathbf{J}_b = \hat{\boldsymbol{\rho}}\left(-\frac{\partial M_\phi}{\partial z}\right) + \hat{\mathbf{z}}\frac{1}{\rho}\left(\frac{\partial\rho M_\phi}{\partial\rho}\right) = \frac{2M_0}{a}\hat{\mathbf{z}} \tag{13.2.1}$$

within the magnet and zero everywhere else. For the surface current densities,

$$\begin{aligned} \mathbf{K}_b &= \mathbf{M}\times\hat{\boldsymbol{\rho}} = -\frac{M_0 b}{a}\hat{\mathbf{z}}, \\ \mathbf{K}_a &= \mathbf{M}\times(-\hat{\boldsymbol{\rho}}) = M_0\hat{\mathbf{z}}, \\ \mathbf{K}_\pm &= \mathbf{M}\times(\pm\hat{\mathbf{z}}) = \pm\frac{M_0\rho}{a}\hat{\boldsymbol{\rho}}, \end{aligned} \tag{13.2.2}$$

where $\mathbf{K}_\pm$ is the surface current density on the two caps. So, there is a current which circulates along the surfaces of the magnet, lengthwise.

As a final note, we point out that the intended problem statement was $\mathbf{M} = \frac{M_0 a}{\rho}\hat{\boldsymbol{\phi}}$ so $\mathbf{J}_b = 0$. Fortunately, this error in problem statement does not actually change anything about the next part of the problem.

13.3 Part (b)

Since the electric fields are not important in this problem, we really only need to solve $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_f + \mu_0 \nabla \times \mathbf{M}$. But since there are no free currents, we may rewrite this as $\nabla \times (\mathbf{B}/\mu_0 - \mathbf{M}) = \nabla \times \mathbf{H} = 0$. Furthermore, $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} = 0$. So, both the curl and divergence of $\mathbf{H}$ vanish. This combined with the requirement that the field strength be zero at infinity implies $\mathbf{H} = 0$ everywhere. It then follows that $\mathbf{B} = \mu_0 \mathbf{M}$, which is only non-zero within the magnet.

14 Problem 13: Electromagnetism

14.1 Problem Statement (Radiation Pressure)

A plane wave with intensity $\langle S \rangle \mathrm{J/m}^2 \cdot \mathrm{s}$ is incident on a totally reflecting, plane surface at an angle θ, where θ is measured relative to the plane normal.

(a) Find the radiation pressure normal to the surface.

(b) Find the total radiation force produced by this plane wave when incident on a perfectly reflecting sphere of radius R.

(c) Find the total radiation force produced by this plane wave when incident on a perfectly absorbing sphere of radius R, and compare with the results of (b).

14.2 Parts (a)-(c)

Recalling that the Poynting vector itself has the units of energy per area per second, dimensional analysis fixes $P = \langle S \rangle / c$ as the radiation pressure. Obviously this is not a derivation of the fact, but a helpful way of remember the formula.

Now, if the light impacts at an angle θ as described, then the portion of area exposed to the light is less than the total by a factor of $\cos\theta$. Furthermore, since we are only interested in the component of the Poynting vector which is normal to the surface, we obtain another factor of $\cos\theta$. There is also an overall factor of 2 to account for the re-emitted photons. So,

$$P = 2\frac{\langle S \rangle}{c} \cos^2\theta. \tag{14.2.1}$$

Now that we have found the pressure normal to the surface. Here, we are interested in computing the force aligned with the propagation direction of the radiation – we know by

symmetry that all other forces will cancel. However, there is yet another cosine associated with picking out this component of the force. So,

$$F = \int \mathrm{d}AP\cos\theta = \frac{2\langle S\rangle}{c} 2\pi R^2 \int_0^1 \cos^3\theta \mathrm{d}(\cos\theta) = \frac{\pi R^2 \langle S\rangle}{c}. \tag{14.2.2}$$

Finally, we mumble something about the cross-section being πR^2 and that, since the sphere is perfectly absorbing, the force must just be $F = PA = \pi R^2 \frac{\langle S\rangle}{c}$ where we have left out the two we originally point in by hand to account for reflections.

I am not a fan of this problem.

15 Problem 14: Electromagnetism

15.1 Problem Statement (Classical Mechanics and Larmor Radiation)

Find the total energy radiated in the head-on collision of a non-relativistic particle of charge q, velocity v_0 against a fixed target of charge Q ($qQ > 0$). Write your result in terms of q,Q, and v_0.

15.2 Solution

There is a very important additional assumption we must make to solve this problem: we must assume that the particle is not self-interacting. The self-interaction force would be given by the Abraham-Lorentz force and is proportional to the jerk of the particle. This is likely not a very good assumption, but if we do not make this assumption, we would be left with an integrao-differential equation to solve. On these exams, it is extremely important that all unstated assumptions necessary to solve the problem must be made. This assumption translates into the statement that

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2(r) + \frac{qQ}{4\pi\epsilon_0 r} \tag{15.2.1}$$

holds for all r.

This also means that the energy radiated away as the particle is incoming is equal to the power radiated as it leaves. Hence, if P is the power radiated, then the radiated energy is

$$E = 2\int_{t_{close}}^{\infty} P\mathrm{d}t = 2\int_R^{\infty} P\frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r, \tag{15.2.2}$$

where R is the radius of closest approach and $\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{v(r)}$. The velocity as a function of the radial distance is then given by (15.2.1). The radiated power is then given by the Larmor formula,

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}. \tag{15.2.3}$$

So, we will also need to compute $a = \frac{\mathrm{d}}{\mathrm{d}t}v = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}r}v = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}v^2(r)$. With this, we find

$$E = 2\frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{qQ}{4\pi\epsilon_0 m}\right)^2 \int_R^\infty \frac{\mathrm{d}r}{r^3}\frac{1}{\sqrt{r\left(v_0^2 r - \frac{qQ}{2\pi m\epsilon_0}\right)}}. \tag{15.2.4}$$

This is a disgusting integral, but it can be evaluated by the substitution $y = \frac{1}{2}mv_0^2 - \frac{Qq}{4\pi\epsilon_0 r}$.