1 Using this Document and Disclaimer

This document is intended to record the solutions to the 2017 comprehensive exam as I understand them. No guarantees are made about the correctness of these solutions, though I have tried to careful to do things correctly. Throughout this document, I have also made an effort to point out how the reader might remember the details common formulae. It is important to note that these explanations are *not* intended to be rigorous justifications, but rather are intended as a collection of mnemonics from which one might hope to interpolate the correct formula from an imperfect memory.

With this said, I do hope this document will prove as useful to others as I hope it will be for myself in creating it. Feedback is appreciated, and should be directed to the author's email: myersr(at_the_system)physics.ucla.edu.

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2 Problem 1: Quantum Mechanics

2.1 Problem Statement (Coherent States)

Consider a 1D harmonic oscillator with creation and annihilation operators $a^\dagger$ and a with the usual commutation relation $[a, a^\dagger] = 1$. An eigenstate of the annihilation operator is called a "coherent state," and is denoted with its eigenvalue β, which is in general a complex number:

$$a|\beta\rangle = \beta|\beta\rangle. \tag{2.1.1}$$

(a) Derive the expansion for a coherent state $|\beta\rangle$ in the basis of number (Fock) states $|n\rangle$, where n is a non-negative integer. Choose a global phase such that $\langle n = 0|\beta\rangle$ is a positive real number. Hint: Recall that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(b) Calculate the overlap integral between two coherent states, $\langle\beta|\alpha\rangle$. Are coherent states with $\alpha \neq \beta$ ever strictly orthogonal?

2.2 Part (a)

Since the collection $\{|n\rangle\}$ forms a complete basis of states, it follows that each vector $|\beta\rangle$ may be expanded in this basis. Hence, there exist some coefficients $C_n^\beta \in \mathbb{C}$ such that

$$|\beta\rangle = \sum_{n=0}^{\infty} C_n^\beta |n\rangle. \tag{2.2.1}$$

But given the defining relation (2.1.1), it follows that these coefficients must satisfy

$$\beta \sum_{n=0}^{\infty} C_n^\beta |n\rangle = \sum_{n=1}^{\infty} \sqrt{n} C_n^\beta |n-1\rangle = \sum_{0}^{\infty} \sqrt{n+1} C_{n+1}^\beta |n\rangle, \tag{2.2.2}$$

from which it follows that

$$C_n^\beta = \frac{\beta}{\sqrt{n}} C_{n-1}^\beta = \frac{\beta^n}{\sqrt{n!}} C_0^\beta, \tag{2.2.3}$$

where the final equality follows by iterating the relation down to $n = 0$ where it terminates. Strictly speaking, this relation should be proved by induction, but will not bother here.

It now follows that the state $|\beta\rangle$ may be written

$$|\beta\rangle = C_0^\beta \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle. \tag{2.2.4}$$

The coefficient C_0^β is therefore the normalization factor of the state. We note also that $\langle n = 0|\beta\rangle = C_0^\beta$, which we were asked to chose real and positive. The normalization of the state in terms of C_0^β is

$$\langle\beta|\beta\rangle = |C_0^\beta|^2 \sum_{n=0}^{\infty} \frac{\beta^{2n}}{n!} = |C_0^\beta|^2 e^{\beta^2}. \tag{2.2.5}$$

This suggests that we choose $C_0^\beta = e^{-\beta^2/2}$ so C_0^β is both real and positive, and we normalize $|\beta\rangle$ to unity. Hence,

$$|\beta\rangle = e^{-\beta^2/2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}}|n\rangle. \tag{2.2.6}$$

2.3 Part (b)

We are now asked to compute the overlap $\langle\beta|\alpha\rangle$. This is the straight forward calculation

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(\beta^2+\alpha^2)} \sum_{n=0}^{\infty} \frac{(\alpha\beta)^2}{n!} = \exp\left[-\frac{1}{2}\left(\alpha^2 - 2\alpha\beta + \beta^2\right)\right] = \exp\left[-\frac{1}{2}(\alpha-\beta)^2\right], \tag{2.3.1}$$

which we note is always strictly greater than zero. This implies that two coherent states are never orthogonal.

3 Problem 2: Quantum Mechanics

3.1 Problem Statement (Bound States/Delta-Well Potential)

Here we study a simple one-dimensional quantum-mechanical system that resembles the hydrogen molecular ion. A particle of mass m is in a potential

$$V(x) = -A_0[\delta(x-a) + \delta(x+a)], \tag{3.1.1}$$

where $A_0 > 0$.

(a) The eigenstates of the (non-relativistic) Hamiltonian for this potential well have definite parity. Without doing any calculation, sketch the amplitude of an even eigenstate as a function of x.

(b) Find the expression that determines the bound-state energy for even-parity states, and determine graphically how many even-parity bound states exist.

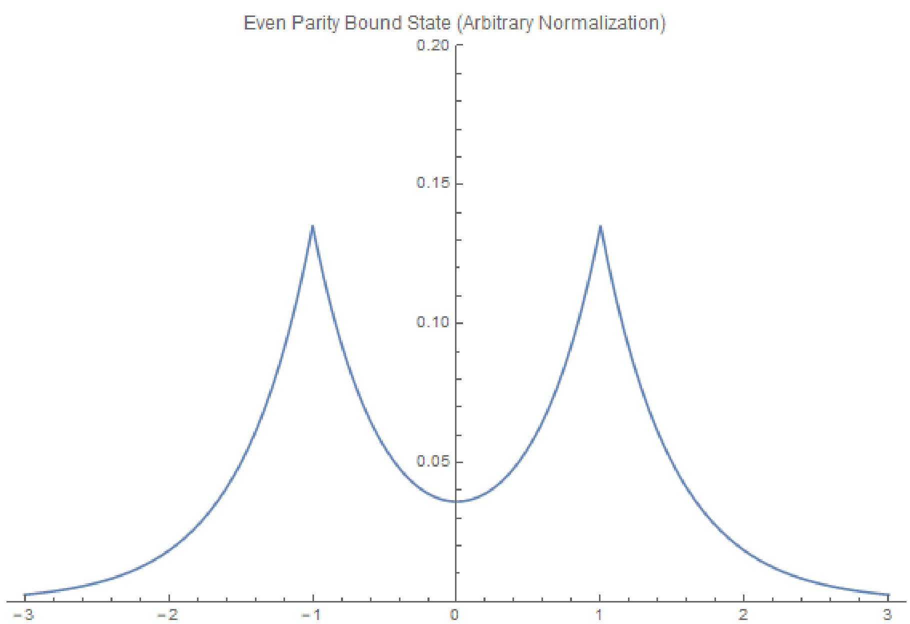
(c) Repeat parts (a) and (b) for odd parity. For what values of A_0 is there at least one such bound state?

Hints: If you take advantage of the symmetry of the problem to minimize the number of constants-to-be-determined in your eigenfunctions (and you should), the constraints on the eigenfunctions at $x = a$ and $x = -a$ will be redundant. You can solve the problem without explicitly normalizing the eigenfunction.

3.2 Part (a)

The correct way to solve this part of the problem is actually to ignore the instructions of the problem and compute the bound state eigenfunctions first, then go back and sketch them. However, if for some reason we cannot find the eigenfunctions first, there are some considerations we could make to get a vague idea what this function should look like.

First, since the amplitude is a parity even function, it is, in fact, and even function of x. So we know the function needs to be symmetric about the origin. Since we are looking for a bound state, we might expect it to be localized around the potential wells, since it is more in keeping with waves that they would be moving and scatter off the potentials instead. So, we could probably take a guess and draw some symmetric function which looks like it's decaying quickly away from the locations of the potential wells. This is a very rough argument, but it might do in a pinch. As I said before, the correct way to do this is to solve the rest of the problem first. We include a plot of the lowest energy bound state below.



3.3 Part (b) - (c)

Since part (c) asks us to solve the problem a second time, but for the odd parity states, we will restrict to parity even states last, to minimize the amount of work we need to replicate for part (c).

The differential equation we must solve is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - A_0[\delta(x-a)+\delta(x+a)]\psi = E\psi \tag{3.3.1}$$

for some E. But E is not actually arbitrary since we are only interested in bound states. For this potential, this means we are interested in $E<0$. If we define the constant $\alpha = \sqrt{\frac{-2mE}{\hbar^2}}$, then for bound states α is real and positive. Then, when we are away from the delta potentials, the differential equation above becomes

$$\frac{\partial^2\psi}{\partial x^2} = \alpha^2\psi, \tag{3.3.2}$$

which clearly has solutions $e^{\pm\alpha x}$. There are three regions of interest – the left, right, and middle – which will get their own solutions which will be some linear combination of the exponential solutions to (3.3.2). Before doing anything else though, we can eliminate the terms which would diverge at infinity for the left and right regions since we demand that wave function be normalizable, and hence must vanish at infinity. This leaves us with

$$\psi_L = Ae^{\alpha x}, \quad \psi_M = Be^{-\alpha x} + Ce^{\alpha x}, \quad \psi_R = De^{-\alpha x}. \tag{3.3.3}$$

Since we are interested in solutions with definite parity, it follows that $A = \pm D$, with the positive sign corresponding to even parity. Parity also implies that ψ_M must be a function of definite parity. Hence, $\psi_M(x) = \pm\psi_M(-x)$, which requires

$$Be^{-\alpha x} + Ce^{\alpha x} = \pm\left(Be^{\alpha x} + Ce^{-\alpha x}\right), \tag{3.3.4}$$

or

$$B\left(e^{-\alpha x} \mp e^{\alpha x}\right) = -C\left(e^{\alpha x} \mp e^{-\alpha x}\right) = \pm C\left(e^{-\alpha x} \mp e^{\alpha x}\right), \tag{3.3.5}$$

so $B = \pm C$. The solution now takes the form

$$\psi_L = \pm De^{\alpha x}, \quad \psi_M = \pm Ce^{-\alpha x} + Ce^{\alpha x} = C\left(e^{\alpha x} \pm e^{-\alpha x}\right), \quad \psi_R = De^{-\alpha x}. \tag{3.3.6}$$

Finally, we need to impose the boundary conditions at $x = a$. We note that since we have constructed a state of definite parity, if we impose the boundary condition at $x = a$, the boundary conditions at $x = -a$ will automatically be satisfied. There are two boundary conditions here. The first is continuity, which simply yields

$$C = D\frac{e^{-\alpha a}}{e^{\alpha a} \pm e^{-\alpha a}}. \tag{3.3.7}$$

We obtain the other boundary condition at $x = a$ by integrating (3.3.1) on an interval of width 2ϵ about $x = a$:

$$\int_{a-\epsilon}^{a+\epsilon} \mathrm{d}x \frac{\partial^2 \psi}{\partial x^2} + \frac{2mA_0}{\hbar^2}\psi(a) = \alpha^2 \int_{a-\epsilon}^{a+\epsilon} \mathrm{d}x \psi \tag{3.3.8}$$

which for $\epsilon \to 0$ implies

$$\frac{\partial \psi_R(a)}{\partial x} - \frac{\partial \psi_M(a)}{\partial x} = -\frac{2mA_0}{\hbar^2}\psi(a). \tag{3.3.9}$$

This implies the condition

$$-\alpha D e^{-\alpha a} - \alpha D e^{-\alpha a}\frac{(e^{\alpha a} \mp e^{-\alpha a})}{(e^{\alpha a} \pm e^{-\alpha a})} = -\frac{2mA_0}{\hbar^2} D e^{-\alpha a}, \tag{3.3.10}$$

which reduces to

$$\alpha \left[1 + \frac{(e^{\alpha a} \mp e^{-\alpha a})}{(e^{\alpha a} \pm e^{-\alpha a})}\right] = \frac{2mA_0}{\hbar^2}. \tag{3.3.11}$$

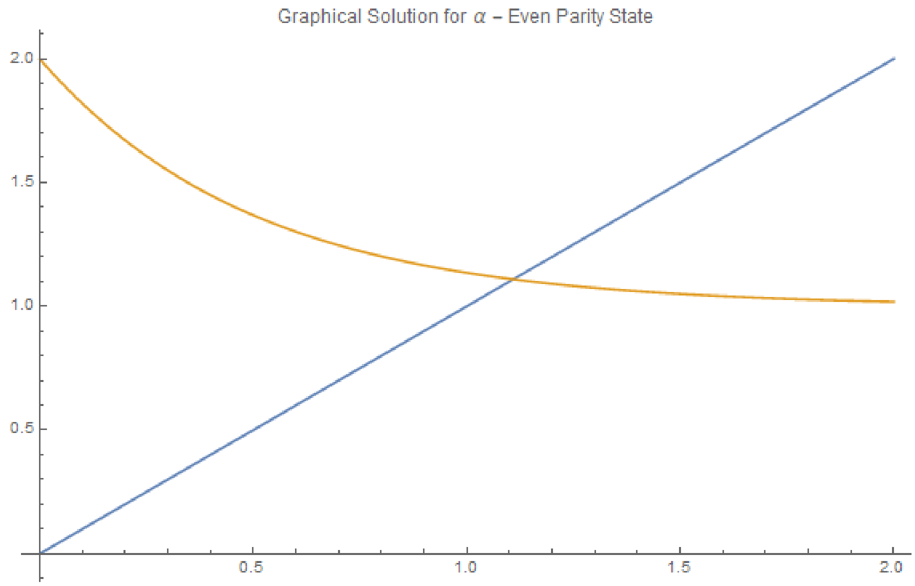
At this point, we have imposed all the boundary conditions and could specialize to either odd or even parity, recognizing the ratio of exponentials as either $\tanh(\alpha a)$ or $1/\tanh(\alpha a)$. However, because we will need to think about solutions for α to this equation, it will be helpful to change the form a bit and write instead,

$$\frac{2\alpha}{1 \pm e^{-2\alpha a}} = \frac{2mA_0}{\hbar^2}, \tag{3.3.12}$$

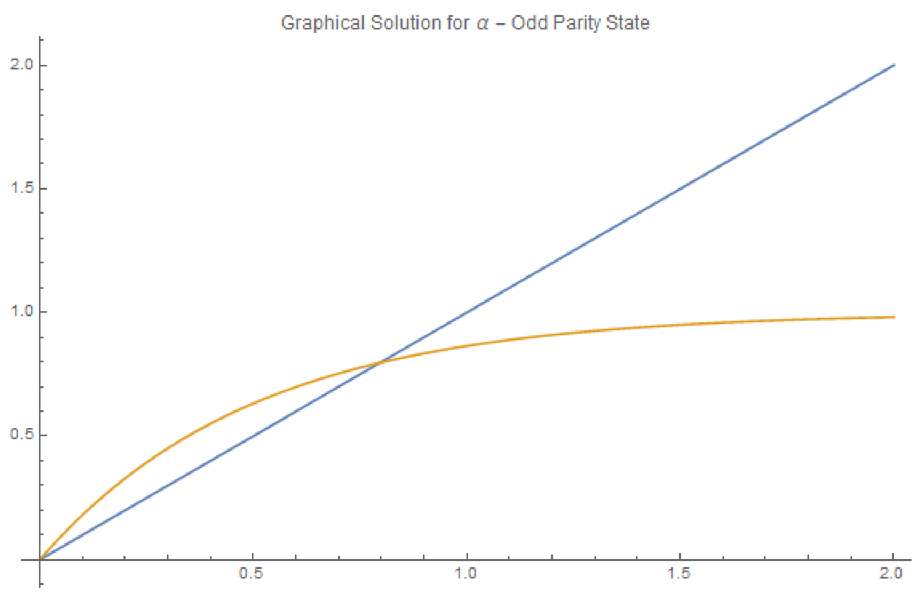
which is then equivalent to

$$\frac{\hbar^2}{mA_0}\alpha = 1 \pm e^{-2\alpha a}. \tag{3.3.13}$$

For even parity now, we take the upper sign and the solution for α is the intersection point of a line passing through the origin with the function $1 + e^{-2\alpha a}$. This exponential function takes the value 2 at $\alpha = 0$, and then decreases monotonically to 1 with increasing α. Hence, we are guaranteed to have exactly one point of intersection between the two curves, and hence exactly one solutions for α, yielding exactly one bound state of even parity.



If we turn to the odd parity state, we choose the lower sign in (3.3.13). On the LHS, we still have a linear function, but on the right hand side, we now have a function which is zero at $\alpha = 0$ and monotonically increasing to 1. Clearly, the two meet at $\alpha = 0$, but we are interested in $\alpha \neq 0$. The only way for these two to intersect is for the slope of the line to be less than the slope of the tangent line to $1 - e^{-2\alpha a}$ at $\alpha = 0$, as can be seen in the graph below. This implies the condition $\frac{\hbar^2}{mA_0} < 2a$ for an odd parity bound state to exist.



4 Problem 3: Quantum Mechanics

4.1 Problem Statement

Consider a spin-half electron in a p-state ($\ell = 1$). The total angular momentum is $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Add explicitly the angular momenta and find the quartet and the doublet in the combined representation in terms of the original representation.

Hint: For any operators satisfying the angular momentum algebra, the raising and lowering operators satisfy

$$J_{\pm}|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle, \tag{4.1.1}$$

where m is the corresponding z-component ($\hbar = 1$). Once you have worked out the quartet states, you should be able to write down the doublet states by inspection.

4.2 Solution

This problem is essentially asking us to compute Clebsch-Gordan coefficients by hand. This is a pain, but as long as we know the algorithm, the problem is just a matter of making no algebraic errors along the way, which is itself a non-trivial task. So, let's discuss the algorithm.

We know from the theory of angular momentum that a product representation of angular momentum states may be reduced to a direct sum of irreducible representations by some change of basis, the change of basis given by the Clebsch-Gordan coefficients. Furthermore, we know that

$$1 \otimes \frac{1}{2} \cong \bigoplus_{k=|1-1/2|}^{|1+1/2|} k = \frac{1}{2} \oplus \frac{3}{2}, \tag{4.2.1}$$

so when we are done, we will have a spin 1/2 irrep. and a spin 3/2 irrep. in the total angular momentum basis.

For the remainder of this problem, we will employ the following notation to keep the calculations more compact. All kets will be labeled by two indices, the first the total angular momenta of the state, and the second the z-component. Furthermore, any time there is a ket by itself, this comes from the total angular momenta basis while whenever we have a tensor product of two states, they will be the product of the spin and angular momenta states – which is which will be determinable from the total angular momentum quantum number of the state.

Since the highest state in the total angular momentum representation is unique, it follows that $|3/2, 3/2\rangle = |1, 1\rangle|1/2, 1/2\rangle$. Using the lowering operator $J_- = J_-^L \oplus 1 + 1 \oplus J_-^S$, we are now able to walk down the lattice of states with $j = 3/2$, of which there are 4. To then get the $J = 1/2$ states, we need to compute the state orthogonal to $|3/2, 1/2\rangle$ with z-component $m = 1/2$. We could then determine $|3/2, -1/2\rangle$ either by the same orthogonality argument or by applying the lowering operator to $|1/2, 1/2\rangle$.

Being careful to make no algebraic errors, we find the $j = 3/2$ states to be

$$\begin{aligned}
|3/2, 3/2\rangle &= |1, 1\rangle|1/2, 1/2\rangle, \\
|3/2, 1/2\rangle &= \frac{1}{\sqrt{3}}\left[\sqrt{2}|1, 0\rangle|1/2, 1/2\rangle + |1, 1\rangle|1/2, -1/2\rangle\right], \\
|3/2, -1/2\rangle &= \frac{1}{\sqrt{3}}\left[|1, -1\rangle|1/2, 1/2\rangle + \sqrt{2}|1, 0\rangle|1/2, -1/2\rangle\right], \\
|3/2, -3/2\rangle &= |1, -1\rangle|1/2, -1/2\rangle.
\end{aligned} \tag{4.2.2}$$

These are the quartet states. To find the doublet states, we need to determine the coefficients $|3/2, 1/2\rangle = a|1, 0\rangle|1/2, 1/2\rangle + b|1, 1\rangle|1/2, -1/2\rangle$ such that the state is orthogonal to $|3/2, 1/2\rangle$ and is normalized. Normalization requires $a^2 + b^2 = 1$ while orthogonality requires $a\sqrt{2} + b = 0$, from which we deduce $b = -a\sqrt{2}$. This then implies $a^2 + 2a^2 = 1$ so $a = \pm 1/\sqrt{3}$. Hence,

$$\begin{aligned}
|1/2, 1/2\rangle &= \frac{1}{\sqrt{3}}\left[\sqrt{2}|1, 1\rangle|1/2, -1/2\rangle - |1, 0\rangle|1/2, 1/2\rangle\right], \\
|1/2, -1/2\rangle &= \frac{1}{\sqrt{3}}\left[|1, 0\rangle|1/2, -1/2\rangle - \sqrt{2}|1, -1\rangle|1/2, 1/2\rangle\right],
\end{aligned} \tag{4.2.3}$$

where we have done the same with $|1/2, -1/2\rangle$. Note that the orthogonality and normalization conditions only ever determine (4.2.3) up to an overall sign. The signs chosen here are conventional. However, the relative sign between these two states is fixed by application of the lowering operator.

5 Problem 4: Quantum Mechanics

5.1 Problem Statement (Stark Effect/Perturbation Theory)

A one-dimensional harmonic oscillator of charge e is perturbed by an electric field E in the positive x-direction. Calculate the change in each energy level to second order in the perturbation, and calculate the induced electric dipole moment. Show that the problem can be solved exactly, and compare the result with the perturbative approximation.

Hint: $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$ acting on harmonic oscillator states $|n\rangle$, where $a^\dagger$ and a are the creation and annihilation operators. Note also the definitions $a = \sqrt{\frac{m\omega}{2\hbar}}(x + ip/m\omega)$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - ip/m\omega)$.

5.2 Solution

Since the potential of the given electric field may be written $\phi = -Ex$, we may invert the relation between the $(a, a^\dagger)$ and (x, p) pairs to write

$$x = \sqrt{\frac{2\hbar}{m\omega}} \frac{a + a^\dagger}{2} \tag{5.2.1}$$

so the electric potential takes the form

$$\phi = -E\sqrt{\frac{2\hbar}{m\omega}} \frac{a + a^\dagger}{2}. \tag{5.2.2}$$

Hence, the Hamiltonian must be

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) - eE\sqrt{\frac{2\hbar}{m\omega}} \frac{a + a^\dagger}{2}. \tag{5.2.3}$$

For the sake of perturbation theory, we now make the definitions

$$H_0 = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right), \qquad V = -eE\sqrt{\frac{2\hbar}{m\omega}} \frac{a + a^\dagger}{2}. \tag{5.2.4}$$

Now, to see that the first order energy correction vanishes, we could argue that the unperturbed eigenstates $|n\rangle$ have definite parity while V is parity odd[1], from which it follows that V has no non-zero diagonal elements. Alternatively, a direct calculation of $\langle n|V|n\rangle$ on the explicit form (5.2.4) would quickly give the same result.

Since the first order perturbation vanishes, we are justified in moving on to the second order correction[2]. We first recall the formula[3]

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)}|V|m^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}. \tag{5.2.5}$$

It is now straightforward to compute $E_n^{(0} - E_m^{(0)} = \hbar\omega(n - m)$ and

$$\langle n^{(0)}|V|m^{(0)}\rangle = -eE\sqrt{\frac{2\hbar}{m\omega}} \frac{1}{2} (\sqrt{m}\langle n^{(0)}|(m-1)^{(0)}\rangle + \sqrt{m+1}|\langle n^{(0)}|(m+1)^{(0)}\rangle). \tag{5.2.6}$$

[1] See the my solution to problem 4 in the 2018 exam for a discussion of parity in the Stark effect.

[2] We note that the problem only actually asks explicitly for the second order correction, so strictly speaking, even if the first order correction didn't vanish, we shouldn't have to compute it. This is, however, a common trick on these exams. You are expected to double check that the question makes sense. We note only that this expectation is antithetical to the format of the comprehensive exam and move on with the reality.

[3] Remember that the state of interest always comes first in this formula.

From here, we just need to put it all together, noting that the cross terms in the square vanish because they are products of Kronecker deltas which are never simultaneously non-zero,

$$\begin{aligned} E_n^{(2)} &= \frac{1}{4}e^2E^2\frac{2\hbar}{m\omega}\frac{1}{\hbar\omega}\sum_{m\neq 0}\left(\frac{m}{n-m}\delta_{n,m-1}+\frac{m+1}{n-m}\delta_{n,m+1}\right) \\ &= \frac{1}{2}\frac{e^2E^2}{m\omega^2}\left(\frac{n+1}{-1}+\frac{n}{1}\right) = -\frac{1}{2}\frac{e^2E^2}{m\omega^2}. \end{aligned} \tag{5.2.7}$$

Our next task is to find the induced dipole moment of the system. Before we embark on this calculation, let's first discuss what this is actually asking us to do. The only place where dipole moment is actually defined is in electromagnetism. There, the definition of the dipole moment is $\mathbf{p} = \int \mathrm{d}V r\rho$. In quantum mechanics, we don't really have a charge density. We are instead only told that the system as a whole has a particular charge. But if we are to believe quantum mechanics that the particle of the system is actually a delocalized wave with some distribution $\psi^\dagger(x)\psi(x)$ in the position basis, then it follows that the charge at a position x is given by $e\psi^\dagger(x)\psi(x)$. This quantity now defines a charge distribution for the system from which we can compute the dipole moment. Using this, the dipole moment becomes $p = \int \mathrm{d}x e\psi^\dagger(x)x\psi(x) = e\langle x\rangle$.

Now, the expectation is basis independent since it only depends on the state of the system we are interested in. Hence, we need to calculate $e\langle n|x|n\rangle$. But we don't actually know how x acts on the exact perturbed states since we have not yet solved the problem exactly. Therefore, we need to express $|n\rangle$ first in terms of the unperturbed basis of states, $|n^{(0)}\rangle$, which requires a first order state perturbation, for which the formula to know is[4]

$$|n^{(1)}\rangle = \sum_{m\neq n}\frac{|m^{(0)}\rangle\langle m^{(0)}|V|n^{(0)}\rangle}{E_n^{(0)}-E_m^{(0)}}. \tag{5.2.8}$$

Fortunately, we have already done all of the computation required for this formula, so we just use our previous results to write

$$|n^{(1)}\rangle = -\frac{1}{2}eE\sqrt{\frac{2}{m\hbar\omega^3}}\left(\sqrt{n}|(n-1)^{(0)}\rangle - \sqrt{n+1}|(n+1)^{(0)}\rangle\right), \tag{5.2.9}$$

so $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \mathcal{O}(E^2)$. Now, computing $e\langle n|x|n\rangle$ is a pain, but if we only keep to

[4]We can remember this formula as being very similar to the second order energy perturbation formula, but first order in V. If we recall vaguely how the derivation of this formula goes, we know the state correction goes like $V|n^{(0)}\rangle$, on which we then insert a complete set of states on the left.

first order in the electric field, it's not the worst. In any case, the end result is

$$e\langle n|x|n\rangle = \frac{e^2E}{m\omega^2}. \tag{5.2.10}$$

To now solve the problem exactly, we first write the Hamiltonian back in terms of the position and momenta operators,

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2 - eEx = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2\left(x - \frac{eE}{m\omega^2}\right)^2 - \frac{e^2E^2}{m^2\omega^4} \tag{5.2.11}$$

where we have completed the square in x. At this point, it makes sense to define the variable $y = x - eE/m\omega^2$. With this definition, the Hamiltonian is again a harmonic oscillator, but now with a constant shift. Defining creation and annihilation operators of this shifted position y, say a_y, the Hamiltonian becomes

$$H = \hbar\omega\left(a_y^\dagger a_y + \frac{1}{2}\right) - \frac{e^2E^2}{m\omega^2}. \tag{5.2.12}$$

Hence, the second order energy perturbation is exact. Furthermore, if we compute

$$e\langle n|x|n\rangle = e\langle n|y + \frac{eE}{m\omega^2}|n\rangle = \frac{e^2E}{m\omega^2}, \tag{5.2.13}$$

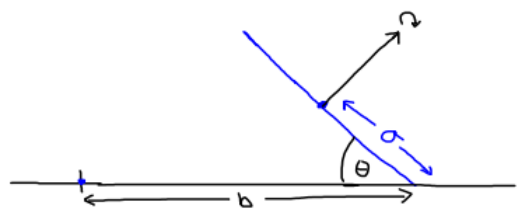
we find that the lowest order perturbation is also exact in the induced dipole moment.

Though that is the end of the problem, it is worth commenting that, though the lowest order perturbation is exact, the perturbation as a whole is likely not correct. Unless the perturbation vanishes at all orders above the lowest, it will not be exact for this problem. So, though it is nice to point out that the lowest order perturbation is exact in this problem, this fact is not helpful when using perturbation in any problem that cannot be solved exactly since there we might want more than just the first correction without knowing any better. If anything, this problem should not be viewed as a warning about putting too much faith in perturbation theory.

6 Problem 5: Classical Mechanics

6.1 Problem Statement (Angular Momentum)

A coin, idealized as a uniform disk of radius a and negligible thickness, is rolling in a circle on a table. The point of contact describes a circle of radius b on the table. The plane of the coin makes an angle θ with the plane of the table. Find the angular velocity ω of the motion of the center of mass of the coin. Hint: you don't need to use a Lagrangian for this problem, just Newton's laws.



6.2 Solution

Since the coin spins about its symmetry axis without slipping on the table, it follows that if ω is the angular velocity of the center of mass and Ω is the angular velocity of the disk, that $a\Omega = b\omega$ in terms of magnitude, clearly the directions are different.

We know that $\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$ and that we are free to decompose $\boldsymbol{\tau} = \boldsymbol{\tau}_{grav} + \boldsymbol{\tau}_{table}$ and $\mathbf{L} = \mathbf{L}_{cm} + \mathbf{L}_{spin}$ where $\mathbf{L}_{cm}$ is the angular momentum of the center of mass about the coordinate center and $\mathbf{L}_{spin}$ is the angular momentum of the disk about its own center of mass.

Since we are already told that the disk is rotating with constant angular velocity, it makes sense to treat this as a two dimensional problem, as is shown in the diagram above. We will then call the vertical direction $\hat{\mathbf{z}}$, the vector pointing to the right $\hat{\mathbf{r}}$, and the vector into the page $\hat{\boldsymbol{\theta}}$. This is obviously a time dependent coordinate system, so if we assume it to align with the static cartesian coordinate system at $t = 0$, we are free to write $\hat{\mathbf{r}} = \cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}$ and $\hat{\boldsymbol{\theta}} = -\sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\hat{\mathbf{y}}$. We call the coordinate center the center of the circle of radius b.

It then follows that[5] $\boldsymbol{\tau}_{grav} = \mathbf{r} \times \mathbf{F} = mg(b - a\cos\theta)\hat{\mathbf{r}} \times (-\hat{\mathbf{z}}) = mg(b - a\cos\theta)\hat{\boldsymbol{\theta}}$. For the torque due to the normal force of the table, we can say a few things. First of all, we know that it's there and that it must have a vertical component along with a horizontal component pointing towards the coordinate center. The vertical component must exist to prevent the disk from falling through the table, and the horizontal component must exist to prevent the disk from falling over.

However, we will not need the horizontal component of the normal force since it supplies no torque, but only the vertical component, which we know must be $mg\hat{\mathbf{z}}$ so the forces in

[5] There is a $\hat{\mathbf{z}}$ component to the leaver arm pointing to the center of mass. However, this component applies no torque.

the z-direction cancel. It then follows that $\boldsymbol{\tau}_{table} = -mgb\hat{\boldsymbol{\theta}}$. Hence, the total torque is

$$\boldsymbol{\tau} = -mga\cos\theta\hat{\boldsymbol{\theta}}. \tag{6.2.1}$$

Next, we need to compute the angular momenta. To start with, we clearly have $\mathbf{L}_{cm} = \mathbf{r}_{cm} \times m(b - a\cos\theta)\omega\hat{\boldsymbol{\theta}}$. Since $\mathbf{r}_{cm} = (b - a\cos\theta)\hat{\mathbf{r}} + a\sin\theta\hat{\mathbf{z}}$, we have

$$\mathbf{L}_{cm} = -am\omega(b - a\cos\theta)\sin\theta\hat{\mathbf{r}} + m\omega(b - a\cos\theta)^2\hat{\mathbf{z}}. \tag{6.2.2}$$

For the spin angular momentum, we have a disk spinning about its center. For this, we will need to moment of inertia of a disk, which we compute to be

$$I = \int \mathrm{d}A\frac{m}{\pi a^2}r^2 = \frac{2m}{a^2}\int_0^a \mathrm{d}r r^3 = \frac{1}{2}ma^2. \tag{6.2.3}$$

Then the spin angular momentum of the disk must be $\mathbf{L}_{spin} = \frac{1}{2}ma^2\boldsymbol{\Omega}$ where $\boldsymbol{\Omega} = -\Omega(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\mathbf{z}}) = -\omega\frac{b}{a}(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\mathbf{z}})$. Thus we find

$$\mathbf{L} = -\, am\omega\left(\frac{3}{2}b - a\cos\theta\right)\sin\theta\hat{\mathbf{r}} + m\omega\left((b - a\cos\theta)^2 - \frac{ab}{2}\cos\theta\right)\hat{\mathbf{z}}. \tag{6.2.4}$$

To compare this with the torque, we need to take a time derivative of $\mathbf{L}$. This means taking a time derivative of $\hat{\mathbf{r}}$. So, we compute

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \omega(-\sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\hat{\mathbf{y}}) = \omega\hat{\boldsymbol{\theta}}. \tag{6.2.5}$$

Thus, $\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$ implies

$$mga\cos\theta = am\omega^2\left(\frac{3}{2}b - a\cos\theta\right)\sin\theta, \tag{6.2.6}$$

which we may now solve for ω to find

$$\omega = \sqrt{\frac{g}{\left(\frac{3}{2}b - a\cos\theta\right)\tan\theta}} = \sqrt{\frac{2g}{(3b - 2a\cos\theta)\tan\theta}}. \tag{6.2.7}$$

7 Problem 6: Classical Mechanics

7.1 Problem Statement (Adiabatic Invariants)

Consider a pendulum consisting of a point mass m attached to a string of slowly increasing length, $\ell(t)$. The motion is confined to a plane and we assume that $|\ell/\dot{\ell}|$ is much greater than the period of oscillation.

(a) Find the Lagrangian $L(\theta, \dot{\theta}, t)$ and the Hamiltonian $H(\theta, p, t)$ of the system.

(b) Is the Hamiltonian H equal to the total energy E of the pendulum? Are E and H conserved?

(c) Derive the equation of motion for θ in the form of a second order ordinary differential equation. When $\dot{\ell} = 0$, what is the frequency of small oscillations?

(d) Show that the amplitude of small oscillations is proportional to $\ell^{-3/4}$ as the length of the string $\ell(t)$ varies. (Hint: Consider using the adiabatic invariant $\int \mathrm{d}\theta p$.)

7.2 Part (a) - (b)

Writing

$$x = \ell \sin\theta, \qquad y = \ell \cos\theta, \tag{7.2.1}$$

we take time derivatives, square, and hence find

$$L = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right) + mg\ell\cos\theta. \tag{7.2.2}$$

This is a fairly standard change of coordinate system, but there is an important point to note here. It is typical to choose the bottom of the pendulum's arc as the reference level for the gravitational potential. In this case, however, this location is time dependent since the length of the pendulum is changing. A more sensible static location to choose for the reference level is then the top of the pendulum, where the pendulum is anchored. In the case where ℓ is a constant, this would just be a constant shift in the Lagrangian, which we know never impacts our dynamics.

With this Lagrangian, we may calculation the momenta,

$$p = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}, \qquad \dot{\theta} = \frac{p}{m\ell^2}. \tag{7.2.3}$$

With this, we find the Hamiltonian to be

$$H = \dot{\theta} p - L = \frac{p^2}{2m\ell^2} - \frac{1}{2} m\dot{\ell}^2 - mg\ell\cos\theta. \tag{7.2.4}$$

Since ℓ depends on time, and H depends on ℓ, it follows that the Hamiltonian is not conserved in general. However, we should note that $\frac{\partial H}{\partial t} = 0$ may be viewed as a differential equation for ℓ. If this differential equation has a solution on shell, then the Hamiltonian will be conserved when the time dependence of ℓ solves this equation. This problem, however, requires us to not think this hard and instead simply answer "no."

As for whether H and E are equal: This is a stupid question. The Hamiltonian is the only meaningful definition for what the word "energy" means in physics. So, by definition of the word energy, yes.

7.3 Part (c)

Since we are asked for a second order differential equation instead of a pair of first order differential equations, we should write down the Euler-Lagrange equations, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$. For the Lagrangian (7.2.2), this is

$$\ddot{\theta} + 2\frac{\dot{\ell}}{\ell}\dot{\theta} = -\frac{g}{\ell}\sin\theta, \tag{7.3.1}$$

where we have manipulated the form so the result is the typical equation for a pendulum plus another term which is proportional to $\dot{\ell}/\ell$, which we are told will be small in this problem.

For this particular part of the question though, we are asked about the case $\dot{\ell} = 0$, so the extra term vanishes. Furthermore, we are told that the oscillations are small, so we are free to expand to first order in θ. This means we employ the usual trick to write $\sin\theta \approx \theta$ and the equations of motion become those of a harmonic oscillator,

$$\ddot{\theta} = -\frac{g}{\ell}\theta \tag{7.3.2}$$

with frequency $\omega = \sqrt{g/\ell}$.

7.4 Part (d)

For this. we just need to approximate the adiabatic invariant $I = \frac{1}{2\pi}\int \mathrm{d}\theta p$ to get the ℓ scaling of the amplitude A of oscillation. Since we are interested in small oscillations and $\dot{\ell}/\ell$ small, the differential equation (7.3.1) may be approximated by (7.3.2). Therefore, the solution is approximately $\theta = Ae^{i\omega t}$.Therefore, we may approximate the derivative by $\dot{\theta} = -i\omega A e^{-i\omega t} = -i\omega\theta$.

Now, to calculate the adiabatic invariant, what we really need is the momenta in terms of θ. The adiabatic invariant is an on-shell quantity, so it is fine to use the solution to the equations of motion in getting the expression for the momentum. Hence, $p = m\ell^2\dot{\theta} \approx m\omega\ell^2\theta$. The integral of interest is therefore

$$I \propto m\omega\ell^2 \int \mathrm{d}\theta\theta \propto m\omega\ell^2 A^2, \tag{7.4.1}$$

where in the last line we have integrated θ over a loop in phase space, so the bounds of the motion were $\pm A$, the sign being irrelevant . All constant numerical factors have been ignored since we are only interested in the scaling. That scaling now tells us

$$A \propto \frac{1}{\ell\sqrt{\omega}} \propto \ell^{-3/4}. \tag{7.4.2}$$

This is the desired amplitude scaling of with ℓ for the system.

8 Problem 7: Electromagnetism

8.1 Problem Statement (Magnetic Scalar Potential)

An infinite straight wire carrying a current I is suspended parallel to the plane interface between vacuum and a medium with magnetic permeability $\mu \neq 1$, at a distance a from the interface. Calculate the force per unit length on the wire, and state whether it is attractive or repulsive.

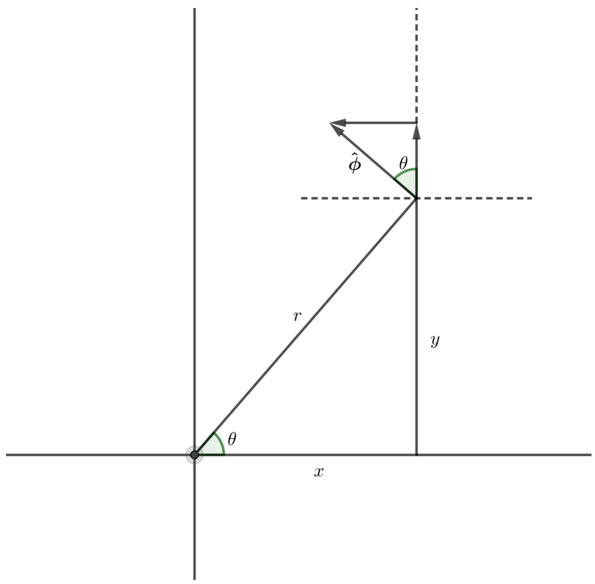
Hint: For this problem, it is helpful to introduce a scalar potential. Also, it is helpful to take coordinates in the complex plane perpendicular to the wire.

8.2 Solution

To begin, we know that the problem is invariant under translations in the z-direction (taken to be along the direction of the current flow, I), so the problem is effectively two-dimensional. In the plane orthogonal to the z-axis, we take $\hat{\mathbf{x}}$ along the surface of the interface plane, and $\hat{\mathbf{y}}$ to be orthogonal to it. Selecting a convenient location for the origin, the wire is at the location $(0, a, z)$ for all z. Since it is ambiguous in the problem statement, we will take the wire to be in the μ_0 region, so the region $y < 0$ has μ.

Since all currents in this problem are constant and there are no static charges, it follows that this is a magnetostatics problem, so we are free to ignore the electric field. If we cared, it would be a simple exercise to show that $\mathbf{E} = 0$.

Defining the field $\mathbf{B} = \mu(y)\mathbf{H}$, we see that $\mathbf{H}$ is a free field, except for the free current in the wire. This means that we may write $\mathbf{H} = \mathbf{H}_I + \nabla f$ within the $y > 0$ region, and $\mathbf{H} = \nabla g$ within the $y < 0$ region, where $\mathbf{H}_I$ is the field due to the wire alone. So, let us first find the free field due to the wire, and then move on from there. If the wire were at the coordinate origin in the xy-plane, we would simply use Ampere's law to write the field as $\frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$. However, we want to shift this formula in the $\hat{\mathbf{y}}$ direction and will need to impose boundary conditions which are rectangular, so it would make sense to first put this expression in rectangular coordinates[6]. With sufficient trigonometry, as shown in the diagram below,



we find the conversion to be

$$\hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}}}{\sqrt{x^2+y^2}} + \frac{x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}. \tag{8.2.1}$$

As we pointed out, however, this is the conversion if the origin were at the wire. We need to shift this conversion up by a, so field due to the wire shifted off the origin is then

$$\mathbf{H}_I = \frac{I\left(-(y-a)\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)}{2\pi(x^2+(y-a)^2)} = \mathbf{H}_I = \frac{I\left((a-y)\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)}{2\pi(x^2+(y-a)^2)}. \tag{8.2.2}$$

Now, we must require that the magnetic field satisfy the boundary conditions $\hat{\mathbf{y}} \cdot (\mathbf{B}_+ - \mathbf{B}_-) = 0$, and $\hat{\mathbf{y}} \times (\mathbf{H}_+ - \mathbf{H}_-) = 0$, where $\pm$ is taken to refer to the sign of y, at the

[6]It is, of course, possible to to the conversion after shifting, but this only serves to make the trigonometry more complicated, and leads to the same result.

surface of the dielectric ($y = 0$). So, the normal component of $\mathbf{B}$ must be continuous across the boundary of the dielectric, and the tangential component of $\mathbf{H}$ must be continuous. Furthermore, this must hold for all x.

Now, completing the problem by using the two dimensional Green's functions for the magnetic scalar potential, we instead suppose that the problem may be solved by images. We set ∇f to be the field due to a wire with current I_2 at $y = -a$ and we set ∇g to be the field due to a wire with current I_3 at $y = a$. It then follows that the boundary conditions at $y = 0$ imply

$$\begin{aligned} \frac{\mu_0 I x}{2\pi(x^2+a^2)} + \frac{\mu_0 I_2 x}{2\pi(x^2+a^2)} &= \frac{\mu I_3 x}{2\pi(x^2+a^2)} \\ \frac{I a}{2\pi(x^2+a^2)} + \frac{-I_2 a}{2\pi(x^2+a^2)} &= \frac{I_3 a}{2\pi(x^2+a^2)}. \end{aligned} \tag{8.2.3}$$

We notice that if we had put the wires at distances from the interface different from a, the denominators would not have canceled, and the condition would not hold for all x. In any case, we now have

$$\begin{aligned} I &= -I_2 + \frac{\mu}{\mu_0} I_3 \\ I &= \quad I_2 + I_3. \end{aligned} \tag{8.2.4}$$

Summing these equations, we find $2I = (1 + \mu/\mu_0) I_3$, which then implies $I_2 = I - I_3$. Therefore,

$$\begin{aligned} I_2 &= \frac{(\mu - \mu_0) I}{\mu + \mu_0}, \\ I_3 &= \frac{2\mu_0 I}{\mu + \mu_0}. \end{aligned} \tag{8.2.5}$$

With this, we now have the magnetic field

$$\begin{aligned} \mathbf{B}_+ &= \frac{(-(y-a)\hat{\mathbf{x}} + x\hat{\mathbf{y}})\mu_0 I}{2\pi(x^2+(y-a)^2)} + \frac{-(y+a)\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{2\pi(x^2+(y+a)^2)} \frac{(\mu-\mu_0)\mu_0 I}{\mu+\mu_0}, \qquad y > 0, \\ \mathbf{B}_- &= \frac{-(y-a)\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{2\pi(x^2+(y-a)^2)} \frac{2\mu\mu_0 I}{\mu+\mu_0}, \qquad y < 0. \end{aligned} \tag{8.2.6}$$

Now that we have the field, we need to compute the force on the wire. This means we will need the field in the $y > 0$ region. Since the magnetic field does not generate any self-forces, the force on the wire must be due to the image wire, whose field is the second term in $\mathbf{B}_+$ above, which we will denote by $\mathbf{B}_{eff}$. By the Lorentz force law, we know that

$\mathrm{d}\mathbf{F} = \mathrm{d}q\mathbf{v} \times \mathbf{B} = I\mathrm{d}\boldsymbol{\ell} \times \mathbf{B}_{eff}$. Evaluated at $(0, a, z)$,

$$\mathbf{B}_{eff} = -\frac{a\hat{\mathbf{x}}}{4\pi a^2}\frac{(\mu - \mu_0)\mu_0 I}{\mu + \mu_0}. \tag{8.2.7}$$

Thus, since $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$, the force per unit length, $\mathbf{f}$, is given by

$$\mathbf{f} = -\frac{\mu_0\hat{\mathbf{y}}}{4\pi a}\frac{\mu - \mu_0}{\mu + \mu_0}I^2. \tag{8.2.8}$$

Hence, when $\mu > \mu_0$, the force is towards the force is towards the interface, and when $\mu < \mu_0$, the force is away from the interface.

9 Problem 8: Electromagnetism

9.1 Problem Statement (Modification to the Free Wave Equation/Klein-Gordon Equation)

The possibility that photons have some small, nonzero rest mass m can be introduced consistently into Maxwell's equations to transform them into the Proca equations:

$$\begin{aligned} &\nabla \cdot \mathbf{E} = \tfrac{\rho}{\epsilon_0} - \tfrac{m^2c^2}{\hbar^2}\phi, \quad && \nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t}, \\ &\nabla \cdot \mathbf{B} = 0, && \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \tfrac{1}{c^2}\tfrac{\partial \mathbf{E}}{\partial t} - \tfrac{m^2c^2}{\hbar^2}\mathbf{A}, \end{aligned} \tag{9.1.1}$$

which are valid for the Lorentz gauge, with the familiar relations given by

$$\begin{aligned} &\nabla \cdot \mathbf{A} = -\tfrac{1}{c^2}\tfrac{\partial \phi}{\partial t}, \quad && \mathbf{B} = \nabla \times \mathbf{A}, \\ &\mathbf{E} = -\nabla\phi - \tfrac{\partial \mathbf{A}}{\partial t}, && c \equiv \tfrac{1}{\sqrt{\epsilon_0\mu_0}}. \end{aligned} \tag{9.1.2}$$

(a) Derive the modified wave equation obeyed by $\mathbf{E}$ in free space (i.e. no charge current density).

(b) Consider transverse electromagnetic plane wave what propagate through free space along the x-axis with electric field magnitude

$$E(x,t) = \frac{1}{2}\varepsilon_0\left(e^{i(kx-\omega t)} + e^{-i(kx-\omega t)}\right). \tag{9.1.3}$$

Derive the expression for the group velocity as a function of optical frequency $\nu_g(\omega) \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k}$ of electromagnetic waves in free space.

Potentially useful vector math: $\nabla \times (\nabla \times \mathbf{G}) = \nabla(\nabla \cdot \mathbf{G}) - \nabla^2\mathbf{G}$.

9.2 Part (a)

Using the given vector identity and taking the curl of $\nabla \times \mathbf{E}$, we find

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} - \frac{m^2c^2}{\hbar^2}\mathbf{A}\right], \tag{9.2.1}$$

where we have used the equation for $\nabla \times \mathbf{B}$. But now, $\nabla(\nabla \cdot \mathbf{E}) = -\frac{m^2c^2}{\hbar^2}\nabla\phi$, so

$$-\frac{m^2c^2}{\hbar^2}\nabla\phi - \nabla^2\mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{m^2c^2}{\hbar^2}\frac{\partial \mathbf{A}}{\partial t}. \tag{9.2.2}$$

Now, since $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$,

$$-\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla^2\mathbf{E} - \frac{m^2c^2}{\hbar^2}\mathbf{E} = 0. \tag{9.2.3}$$

This equation is known as the Klein-Gordon equation, and is typically written in natural units and covariant notation as $(\partial^2 - m^2)\mathbf{E} = 0$, where we have used the $(-,+,+,+)$ signature, which is the Lorentz invariant wave equation for a massive particle.

9.3 Part (b)

Since the wave equation we have found is linear in $\mathbf{E}$, the exact form (9.1.3) of the electric field is irrelevant for finding the dispersion relation, $\omega(k)$, from which we derive the group velocity. It only maters that $\mathbf{E} \propto e^{i(kx-\omega t)}$. Alternatively, we could just Fourier transform the wave equation. Either way, the result amounts to the same:

$$\frac{1}{c^2}\omega^2 - k^2 - \frac{m^2c^2}{\hbar^2} = 0, \tag{9.3.1}$$

so we find

$$\omega^2 = c^2k^2 + \frac{m^2c^4}{\hbar^2} \tag{9.3.2}$$

for our dispersion relation. Simply taking the k derivative of ω, we find

$$v_g(\omega, k) = \frac{c^2k}{\omega}. \tag{9.3.3}$$

But since the question wants the velocity as a function of only ω, we must write $k = \frac{1}{c}\sqrt{\omega^2 - \frac{m^2c^4}{\hbar^2}}$. Hence,

$$v_g(\omega) = \frac{c}{\omega}\sqrt{\omega^2 - \frac{m^2c^4}{\hbar^2}} = c\sqrt{1 - \left(\frac{mc^2}{\hbar\omega}\right)^2}. \tag{9.3.4}$$

10 Problem 9: Electromagnetism

10.1 Problem Statement (Induced Currents and Forces)

(a) An element of wire of oriented length $\mathrm{d}\boldsymbol{\ell}$ is moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$. Starting from the Lorentz force law, calculate the motional EMF $\mathrm{d}\varepsilon$ developed in the element.

(b) A large sheet of copper moves with constant velocity $\mathbf{v}$ through the narrow gap of a C-shaped permanent magnet. The copper has thickness h and conductivity σ. The magnet's field may be considered to have a constant value B_0 inside, and be negligible outside, the rectangular area $w \times \ell$ determined by the magnet's pole pieces (take the sheet's velocity to be parallel to the w dimension). The motion indices an EMF in the conducting sheet, which drives a two-dimensional pattern of eddy currents in the sheet. The portion of this current pattern flowing within the region of magnetic field experiences a Lorentz force. Calculate, to within a constant of proportionality α, the resulting electromagnetic force on the moving sheet (state explicitly the direction of this force).

The dimensionless constant α is of order unity and is meant to save you the trouble of calculating the exact path of the eddy currents in the sheet. Take α to be the constant of proportionality relating the total resistance of the current path (which is difficult to calculate) to the resistance of a piece of the current path that you can calculate easily. However, you decide to define α, be sure to state your definition clearly.

10.2 Part (a)

For this problem, we have to remember that the motional EMF is defined to be $\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$ where Φ_B is the magnetic flux. We are interested in the EMF across a line element $\mathrm{d}\boldsymbol{\ell}$ which has velocity $\mathbf{v}$. By carefully inspecting thine right hand, the Lorentz force law tells us that only the component of the magnetic field in the direction of the oriented surface swept out by the line element will induce an EMF. Then over a small interval of time, say $\mathrm{d}t$, the line element will have swept out an area $\mathbf{v} \times \mathrm{d}\boldsymbol{\ell}\mathrm{d}t$, so the flux through this region is $\mathrm{d}\Phi_B = \mathbf{B} \cdot (\mathbf{v} \times \mathrm{d}\boldsymbol{\ell})\mathrm{d}t$, which then gives us the time change in magnetic flux to be

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = (\mathbf{v} \times \mathrm{d}\boldsymbol{\ell}) \cdot \mathbf{B}. \tag{10.2.1}$$

It therefore follows that $\mathrm{d}\varepsilon = -(\mathbf{v} \times \mathrm{d}\boldsymbol{\ell}) \cdot \mathbf{B}$.

Now, if you do not like the iffy argument that essentially amounts to asserting that the area element of a line element is well-defined, we could note that by the Lorentz force law, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ for this problem and then the work done to move a particle across the element would be $W = \mathbf{F} \cdot \mathrm{d}\ell$. Since the EMF may be thought of as the work per charge, $\mathrm{d}\varepsilon = (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\boldsymbol{\ell}$. This expression is, in fact, equivalent to the expression we found above for $\mathrm{d}\varepsilon$. The simplest way to see this is to note that for any three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is invariant under cyclic permutations of the three vectors and then use the antisymmetry of the cross product. The cyclicity property follows from the antisymmetry of the Levi-Civita symbol.

10.3 Part (b)

Now, the wording on this problem statement isn't the best, but the important thing is that there is a sheet of some dimensions and the particular dimensions that are exposed to magnetic field are $w \times \ell \times h$. The magnetic field is orthogonal to the $w \times \ell$ plane, and the velocity is along the same direction as the side of the rectangle with has dimension w.

Since $\mathrm{d}\varepsilon = -\mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell}$, we can use the result of part (a) to relate the electric field to the velocity and magnetic field. As is alluded to in the problem statement though, we would need to take $\mathrm{d}\boldsymbol{\ell}$ along the lines of current in the sheet. We don't know what these curves should be, so we can't do an exact calculation[7]. This is fine though, since the constant α will be what accounts for the unknown geometry.

Sheets of copper are conductors, so we know $\mathbf{J} = \sigma\mathbf{E}$ for some constant σ. We will be interested in computing the current $I = \int \mathbf{J} \cdot \mathrm{d}\mathbf{A}$ with the integration over the cross-sectional area of the sheet orthogonal[8] to $\mathbf{v}$. This is a slice of the rectangular prism with dimensions $\ell \times w$. Since we don't know how the direction of $\mathbf{J}$ varies with position in the cross-section, we let α be the geometric factor which accounts for this. That is, we write $I = -\alpha\sigma v B h \ell$.

Now, since $\mathrm{d}\mathbf{F} = I(\mathrm{d}\mathbf{w} \times \mathbf{B})$ where the differential length is taken over the dimension w since the current we calculated is parallel to this dimension. It follows that if the magnetic field is in the $\hat{\mathbf{z}}$ direction and $\mathbf{v}$ is in the $\hat{\mathbf{x}}$ direction, then

$$\mathbf{F} = -IwB\hat{\mathbf{y}} = \alpha\sigma v B^2 (wh\ell)\hat{\mathbf{y}}, \tag{10.3.1}$$

where $\hat{\mathbf{y}}$ points in the direction that makes (x, y, z) a right-handed coordinate system. That is, $\hat{\mathbf{y}}$ points in the port direction of the sheet's potion so $-\hat{\mathbf{y}}$ points to the starboard.

[7] I believe the task of finding the exact current density within the sheet would require us to solve Maxwell's equations in differential form.

[8] We might worry that it matters which particular slice we choose when integrating, but by conservation of charge, we know that the current must be the same in all slices within the magnet.

11 Problem 10: Electromagnetism

11.1 Problem Statement (Relativistic Particle-Particle Scattering)

Consider the backscattering of laser photons from a counter-propagating relativistic electron. The electron is taken to be traveling in the z-direction with speed v giving a Lorentz factor $\gamma = \sqrt{1-(v/c)^2}$. The laser photons propagate in the opposite $(-\hat{\mathbf{z}})$ direction and the scattered photons in the positive z-direction. The laser photons have a free-space wavelength of 800 nm ($\hbar\omega = 1.55$ eV), and the electron total energy is 100 MeV.

(a) Assuming the *Thomson* approximation, the frequency ω' of the scattered radiation in the *electron rest frame* obeys $\hbar\omega' \ll m_e c^2$, and the backscattered radiation has nearly the same frequency, but opposite wavenumber k' (reversed propagation direction) as the laser in this frame. Write expressions for k' and ω', and evaluate the adequacy of the Thomson approximation.

(b) What is the energy of the scattered photons in the laboratory frame?

11.2 Part (a)

The wording of this problem is a bit odd, but to be clear, we are *not* asked to make the Thomson approximation. We are instead asked to do the problem in detail, and then show that the Thomson approximation is a good one for $\hbar\omega' \ll m_e c^2$. We note that this is *not* a wave-scattering problem. Instead, we should treat the photons as particles for the purposes of the scattering. Furthermore, the statement is somewhat ambiguous when it comes to what frame the frequencies should be evaluated in. We will take ω and k to be the frequency and wave numbers of the photon in the electron rest frame before the collision, and ω' and k' to be the frequency and wave numbers for the scattered photon *still in the same frame.* Generally, we will refer to post-scattering quantities with primes.

So, in the electron rest frame, $P_e = (m_e c, 0, 0, 0)$, $P_p = \left(\frac{\hbar\omega}{c}, 0, 0, -\hbar k\right)$, where we have taken the signature $(-,+,+,+)$. If we did not recall the form of the 4-momenta for a photon, we can instead just recall that the energy of a photon is given by $\hbar\omega$, and that $P_p^2 = 0$ since the photon is massless, along with the dispersion relation for free photons. That is, $0 = P_p^2 = -\frac{\hbar^2\omega^2}{c^2} - x^2$ for some unknown x^2. But then since $\omega = ck$, $x = -\hbar k$, where the sign is fixed by recalling that the direction of the incoming photon is fixed to be the $-\hat{\mathbf{z}}$ direction.

We are told that after the collision, the photon moves in the $\hat{\mathbf{z}}$ direction, so it must have some momenta $P_p' = \left(\frac{\hbar\omega'}{c}, 0, 0, \hbar k\right)$. Meanwhile, the electron has some post-collision momenta, $P_e' = (E'/c, 0, 0, p_e')$.

Then by the conservation of momentum[9], $P_e + P_p = P_e' + P_p'$, which then implies $(P_e - P_e')^2 = (P_p - P_p')^2$. Using $P^2 = -m^2c^2$ for 4-momenta, independent of the reference frame, we obtain $E^2 = m^2c^4 + c^2p^2$ where lower case p is the three momenta of the particle in question. Furthermore, this allows us to write

$$-2m_e^2c^2 - 2P_e \cdot P_e' = -2P_p \cdot P_p'. \tag{11.2.1}$$

But $P_p \cdot P_p' = \hbar^2(-\omega\omega'/c^2 + kk')$ and $P_e \cdot P_e' = -EE'/c^2 + \mathbf{p}_e \cdot \mathbf{p}_e' = -m_e\sqrt{m_e^2c^4 + c^2{p_e'}^2}$. Therefore, we find

$$-m_e^2c^2 + m_e\sqrt{m_e^2c^4 + c^2{p_e'}^2} = \frac{\hbar^2}{c^2}(\omega\omega' + c^2kk'). \tag{11.2.2}$$

We can always convert the wave numbers to frequencies by $\omega = ck$, so the last thing we really need to do here before we can solve for ω' is find an expression for p_e'. The easiest way to do this really is just to note that $-\hbar k = \hbar k + p_e'$, so ${p_e'}^2 = \hbar^2(k + k')^2$. Hence,

$$-m_e^2c^2 + m_e\sqrt{m_e^2c^4 + c^2\hbar^2(k + k')^2} = \frac{\hbar^2}{c^2}(\omega\omega' + c^2kk'), \tag{11.2.3}$$

with which we can then use the dispersion relation to write

$$\sqrt{m_e^2c^2 + \frac{\hbar^2}{c^2}(\omega + \omega')^2} = m_ec\left(1 + 2\frac{\hbar^2}{m_e^2c^4}\omega\omega'\right). \tag{11.2.4}$$

Squaring and then collecting terms to be polynomial in ω',

$$\left(1 - 4\frac{\hbar^2}{m^2c^4}\omega^2\right)\omega'^{\,2} - 2\omega\omega' + \omega^2 = 0, \tag{11.2.5}$$

which then admits solutions

$$\omega' = \omega\left[\frac{1 \pm \frac{2\hbar}{m_ec^2}\omega}{1 - \left(\frac{2\hbar}{m_ec^2}\omega\right)^2}\right] = \frac{\omega}{1 \mp \frac{2\hbar}{m_ec^2}\omega}. \tag{11.2.6}$$

We also see that in the limit $\hbar\omega' \ll m_ec^2$, we clearly have $\omega' \approx \omega$.

For the particular numbers given, since the rest mass of an electron is about 0.5 MeV, the first order correction in (11.2.6) is $\mathcal{O}\left(\frac{\hbar\omega}{m_ec^2}\right) = \mathcal{O}\left(\frac{1.55\text{ eV}}{0.5\text{ MeV}}\right) = \mathcal{O}(10^{-6})$ as a fraction of the initial frequency. This means the correction is about 10^{-3} nm on an 800 nm initial frequency. The approximation is likely good enough.

[9] Since we already know the forms of the momenta, we could just impose conservation component by component. However, we will look to do this by relying more on frame-independent arguments with the hope that the technique will be more applicable in other problems.

11.3 Part (b)

If you are like me and can never remember the canned formula for the Doppler shift, we can instead recall the matrix form of the Lorentz boost along a single coordinate axis (I typically remember the boost along the x-axis) and just boost the 4-momenta of the scattered photon. Thus,

$$L(-\beta)P_p' = \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} \hbar\omega'/c \\ \hbar k' \end{bmatrix} = \gamma \begin{bmatrix} \hbar\omega'/c + \hbar\beta k' \\ \hbar\beta\omega'/c + \hbar k' \end{bmatrix}, \tag{11.3.1}$$

from which we deduce that the scattered frequency in the moving frame must be $\omega'' = \omega'(1+\beta)\gamma$. If we were interested in the standard Doppler shift formula, we need only recall that $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{(1+\beta)(1-\beta)}$, so

$$\omega'' = \omega'\sqrt{\frac{1+\beta}{1-\beta}}. \tag{11.3.2}$$

12 Problem 11: Statistical Mechanics

12.1 Problem Statement (Noise Modeling of an LC Circuit)

Consider a closed LC circuit. It is to be used as a thermometer by measuring the rms voltage across the capacitance (and inductance). Find an expression for the temperature dependence of the rms voltage, valid for all temperatures. Then find the limits for high and low temperature.

Hint: at low enough temperatures, quantum effects may be important.

12.2 Solution

From this problem statement, it is not clear how the problem expects us to model the situation. However, we could imagine that the LC circuit is in a box and we have no information about the state of the system, except perhaps an estimate of the average energy. It would then follow that we should consider the canonical ensemble of the LC circuit system.

We recall by Kirchhoff's laws, the equation of motion for the LC circuit is

$$L\frac{\mathrm{d}I}{\mathrm{d}t} - \frac{1}{C}Q = 0, \tag{12.2.1}$$

where L is the inductance of the inductor, and C is the capacitance of the capacitor. Since $\dot{I} = \ddot{Q}$, it follows that this is the equation of a harmonic oscillator with mass L, spring constant $1/C$, and hence frequency $\omega = 1/\sqrt{LC}$. We then know the Lagrangian for this system is $L = \frac{1}{2}L\dot{Q}^2 - \frac{1}{2C}Q^2$ and the Hamiltonian is

$$H = \frac{1}{2L}P^2 + \frac{1}{2}L\omega^2 Q^2. \tag{12.2.2}$$

We should interpret the question's hint that "quantum effects" might be important to mean that we should quantize the above Hamiltonian and view it as an operator on some Hilbert space. It will be convenient to perform the usual change of variables to the creation and annihilation operators, $H = \hbar\omega(a^\dagger a + 1/2)$.

With this, the partition function for the system in the canonical ensemble is given by

$$Z = \mathrm{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} = e^{-\beta\hbar\omega/2}\frac{1}{1-e^{-\beta\hbar\omega}} = \frac{2}{\sinh(\beta\hbar\omega/2)}. \tag{12.2.3}$$

Now, we are interested in computing $\sqrt{\langle V^2\rangle}$, for which it would be sufficient to compute $\langle V^2\rangle$. But since $V = Q/C$, we actually need only compute $\langle Q^2\rangle$. But

$$\langle Q^2\rangle = -\frac{1}{\omega\beta L}\frac{1}{Z}\frac{\partial}{\partial\omega}Z = \frac{\hbar}{2\omega L}\frac{1}{\tanh(\beta\hbar\omega/2)}. \tag{12.2.4}$$

An important take-away from this problem is that it is often convenient to perform a canonical transform on the system to compute the partition function. But since the trace, and hence the partition function, is coordinate invariant, we can use whatever coordinate description of the Hamiltonian is most convenient when deciding what derivative we need to take. In this problem, the number basis was most convenient for computing the trace, but when it came time to compute $\langle Q^2\rangle$, it was more convenient to consider original coordinates (12.2.2) for the Hamiltonian and then take a derivative with respect to ω (and book-keeping the extra stuff that comes along with the derivative that we don't want).

As for the limiting cases, we are interested in high temperature ($\beta \to 0$) and low temperature ($\beta \to \infty$). Since the only temperature dependence comes from the tanh, we first need to know the limiting behavior of the tanh. So,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \begin{cases} 1, & x \to \infty. \\ x, & x \to 0. \end{cases} \tag{12.2.5}$$

So, for low temperature, $V_{rms} \to \sqrt{\frac{\hbar\omega}{2C}}$ and for high temperature, $V_{rms} \to \sqrt{\frac{\hbar\omega}{2C}\frac{2}{\beta\hbar\omega}} = \sqrt{\frac{kT}{C}}$.

13 Problem 12: Statistical Mechanics

13.1 Problem Statement (Modeling Polymer Chains)

(a) Consider a polymer chain comprising n segments. Each segment of length d can freely rotate relative to each other. The system is in 3D and at temperature T. What is the mean displacement $\langle r_{1N} \rangle$ between the chain ends?

(b) Apply a stretching force, f, to both ends of the polymer. Find the partition function as a function of f and then use it to derive the mean distance between the ends.

(c) Now, the polymer in (a) is suspended at one end in a gravitational field of strength g. The mass of each segment of the chain is m. Calculate the average length of the chain. (Hint: you can replace a sum by an integral when N is large.)

13.2 Part (a)

This problem is, again, poorly written. We are supposed to assume that each end of the link is affixed to one *particular* other link, and as it rotates, it drags the end of the attached link with it. In this way, the orientations of the links are able to cancel each other. This is in opposition to the other possible reading of this problem, which involves fixing the links to be aligned linearly, so the effective length of a given link is given by $|\cos\phi|$. Under the expected interpretation, we instead need to compute the expected orientation vector, which is now a signed quantity, which allows cancellations. As we will see, this cancellation will be catastrophic.

Suppose for the moment that each link has a mass m. Furthermore, we will define $a = d/2$, which is then the distance from the center of the link to the end of the link. Then by applying the coordinate transformation $x = a\cos\theta\sin\phi, y = a\sin\theta\sin\phi$, and $z = a\cos\phi$ to compute the usual kinetic energy, then transform the free Lagrangian for a single link to the Hamiltonian description, we find[10]

$$H_1 = \frac{P_\theta^2}{2ma^2\sin^2\phi} + \frac{P_\phi^2}{2ma^2}. \tag{13.2.1}$$

The probability distribution in the canonical ensemble is given by $P_1(P_\theta, P_\phi, \theta, \phi) = \frac{1}{Z_1}e^{-\beta H_1}$. To be more precise, this is the distribution for a single link. Since the links are non-interacting and the Hamiltonian for the full system factors, the entire probability distribution factors (up to a division by $N!$ to remove overcounting). It follows that the marginal distribution

[10] By the way, this is also a good way to remember the Laplacian in spherical coordinates.

for a single particle is just P_1 above. Since the expectation will be the same for every link, $\langle \hat{r}_{1N} \rangle = N \langle \hat{r}_{12} \rangle$. So, we just need to compute the expectation for a single link, then multiply it by Nd. We compute

$$\langle \hat{r}_{12} \rangle = \frac{1}{Z_1} \int \frac{\mathrm{d}P_\theta \mathrm{d}P_\phi}{(2\pi\hbar)^2} \mathrm{d}\theta \mathrm{d}\phi e^{-\beta H} \hat{r}_{12}(\theta, \phi) = \frac{1}{Z_1} \frac{2\pi m}{\beta(2\pi\hbar)^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi |\sin\phi| \hat{r}_{12}(\theta, \phi), \tag{13.2.2}$$

where we have already done the momenta integrals. So, we just need the function $\hat{r}_{12}$.

But this is just the unit vector we described at the beginning of the solution, $\hat{r}_{12} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. So, we notice for the x and y components, we are integrating either the sine of cosine of θ over a full period, which is zero. For the z component, we notice that $|\sin\phi| \cos\phi$ is an odd function about the point $\phi = \pi/2$, so that integral is zero as well. Hence, we find $\langle r_{1,N} \rangle = 0$.

13.3 Part (b)

There is now an energy associated with the links, so they are no longer free, though they are still non-interacting and hence we can rely on the partition function to factor. The energy now associated with the links is $-fd\cos\alpha$ for some angle α. Without loss of generality, we may choose this angle to be the angle ϕ. Then computing the single link partition function (doing the momenta integrals immediately),

$$\begin{aligned} Z_1 &= \frac{2\pi m}{\beta(2\pi\hbar)^2} \int_0^{2\pi} \mathrm{d}\theta \int_{-1}^{1} \mathrm{d}(\cos\phi) e^{\beta f d \cos\phi} = \frac{m}{\beta\hbar^2} \left[\frac{1}{\beta f d} e^{\beta f d \cos\phi} \right]_{-1}^{1} \\ &= \frac{2m}{\beta\hbar^2} \frac{\sinh(\beta f d)}{\beta f d} = \frac{2m}{f d \beta^2 \hbar^2} \sinh(\beta f d). \end{aligned} \tag{13.3.1}$$

We note that $\langle d\cos\phi \rangle = \frac{1}{\beta}\frac{\partial}{\partial f} \ln Z_1$, so

$$\langle d\cos\phi \rangle = -\frac{1}{\beta f} + \frac{d}{\tanh(\beta f d)}. \tag{13.3.2}$$

We find the total length by multiplying this by N,

$$\langle r_{1N} \rangle = N \left(-\frac{1}{\beta f} + \frac{d}{\tanh(\beta f d)} \right). \tag{13.3.3}$$

13.4 Part (c)

Building on the work we did in part (b), we again have a force on each link in the chain. The difference now, however, is that every link has a slightly different energy, since the force on each link will be the weight of the links below it. So, if we index the links from the bottom of the chain, link number n will have n links below it (zero indexing). This means that the force on the n^{th} link will be $f_n = mgn$.

The displacement on the n^{th} link is still given by (13.3.2), but since the f is not the same between all links, we now need to explicitly sum over the link indices:

$$\langle r_{1N} \rangle = \frac{1}{\beta} \sum_{n=0}^{N} \frac{\partial}{\partial f} \ln Z_n \approx \frac{1}{\beta} \int_0^N \mathrm{d}n \frac{\partial}{\partial f} \ln Z_n = \frac{1}{\beta} \int_0^N \frac{\partial n}{\partial f} \frac{\partial}{\partial n} \ln Z_n. \tag{13.4.1}$$

Since $n = f/mg$, we have $\frac{\partial n}{\partial f} = 1/mg$. Then

$$\langle r_{1n} \rangle \approx \frac{1}{mg\beta} \ln Z_n \bigg|_0^N = \frac{1}{mg\beta} \ln \frac{Z_N}{Z_0} = \frac{1}{mg\beta} \ln \frac{\sinh(mgN\beta d)}{mgN\beta d}, \tag{13.4.2}$$

where we have used

$$\lim_{x \to 0} \frac{\sinh(x)}{x} = 0. \tag{13.4.3}$$