1 Using this Document and Disclaimer

This document is intended to record the solutions to the 2018 comprehensive exam as I understand them. No guarantees are made about the correctness of these solutions[1], though I have tried to careful to do things correctly. Throughout this document, I have also made an effort to point out how the reader might remember the details common formulae. It is important to note that these explanations are *not* intended to be rigorous justifications, but rather are intended as a collection of mnemonics from which one might hope to interpolate the correct formula from an imperfect memory.

With this said, I do hope this document will prove as useful to others as I hope it will be for myself in creating it. Feedback is appreciated, and should be directed to the author's email: myersr(at_the_system)physics.ucla.edu.

Contents

[1] An up to date version will be kept at https://www.richardmmyers.com/education.

2 Problem 1: Quantum Mechanics

2.1 Problem Statement (Constant Background Magnetic Field)

A particle of charge q is subjected to a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) Consider the symmetric gauge for the vector potential

$$\mathbf{A} = \frac{1}{2}B\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right) \tag{2.1.1}$$

and show that it correctly gives the magnetic field. Write down the Hamiltonian in the symmetric gauge and define

$$Q = \frac{1}{qB}(cp_x + qyB/2), \qquad P = (p_y - qBx/2c). \tag{2.1.2}$$

Show that the commutator $[Q, P] = i\hbar$. Take c to be the speed of light.

(b) Show that H in terms of P and Q becomes a one-dimensional harmonic oscillator problem, where $\omega = qB/mc$. Find the energy eigenvalues.

(c) Write down the annihilation operator, a, for this harmonic oscillator in terms of the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. Show that the ground state wave function takes the form $\psi_0(z, \bar{z}) = u(z, \bar{z})\exp[-qBz\bar{z}/2\hbar c]$, where u is an arbitrary analytic function $\bar{\partial}u(z, \bar{z}) = 0$.

Hint: The Cauchy-Riemann conditions for the analyticity of a function $f = U(x, y) + iV(x, y)$, $U, V \in \mathbb{R}$ are

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \qquad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}. \tag{2.1.3}$$

2.2 Part (a)

We first check that the given vector potential produces the desired magnetic field. Recall that $\mathbf{B} \equiv \nabla \times \mathbf{A}$. Then we compute

$$B = (\partial_x A_y - \partial_y A_x)\hat{\mathbf{z}} + (\partial_y A_z - \partial_z A_y)\hat{\mathbf{x}} + (\partial_z A_x - \partial_x A_z)\hat{\mathbf{y}} = B\hat{\mathbf{z}}, \tag{2.2.1}$$

as desired. We can remember the Cartesian curl as cyclic permutations of (x, y, z). There is also the determinant formula:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}. \tag{2.2.2}$$

In any case, we have shown that the given vector potential is correct. Next, we write down the Hamiltonian[2]

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 = \frac{1}{2m}\left[\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2 + p_z^2\right]. \tag{2.2.3}$$

We will discuss this Hamiltonian in more detail in problem 6 of this comprehensive exam. So, we now just need to show that $[Q, P] = i\hbar$. Using the linearity of the commutator, we have

$$\begin{aligned}[Q, P] &= \left[\frac{1}{qB}(cp_x + qyB/2), (p_y - qBx/2c)\right] \\ &= \frac{c}{qB}[p_x, p_y] - \frac{1}{2}[p_x, x] + \frac{1}{2}[y, p_y] - \frac{1}{2c}[y, x] = \frac{1}{2}\left([x, p_x] + [y, p_y]\right) = i\hbar,\end{aligned} \tag{2.2.4}$$

as desired.

2.3 Part (b)

Now, if we look at the Hamiltonian (2.2.3), we notice that the first two terms are exactly Q^2 and P^2 up to some overall constants. So,

$$H = \frac{1}{2m}\left[\frac{q^2B^2}{c^2}Q^2 + P^2 + p_z^2\right] = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2Q^2 + \frac{1}{2m}p_z^2, \tag{2.3.1}$$

where we have used the definition $\omega = qB/mc$ given in the problem statement. If we recall that the Hamiltonian may be written in terms of the number operator[3], $n = a^\dagger a$, as $H = \hbar\omega(n + 1/2)$, then the energy eigenvalues of the harmonic part of the Hamiltonian must be $E_n = \hbar\omega(n + 1/2)$. However, we cannot forget about the z-momenta. Its presence means that n alone is not sufficient to index the energy states. So, we we may index the z-momenta by the wave number k so the energies are

$$E_{n,k} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{k^2}{2m}. \tag{2.3.2}$$

For the remainder of the problem, however, the z-momenta are irrelevant.

[2]Unfortunately, the author of this problem decided to use Gaussian units instead of SI without saying so. This can cause some issues with missing factors of c. In this solution we have tracked the factors carefully, but on the exam setting $c = 1$ fixes a fair few issues of this type (up to factors of 4π).

[3]We note that $a^\dagger$ is the creation operator because the † is like a plus, for creation. Furthermore, we know $n = a^\dagger a$ rather than the other way around because we know the vacuum state $|0\rangle$ must have $n|0\rangle = 0$. This is true for $n = a^\dagger a$ and is false for $aa^\dagger$.

2.4 Part (c)

Before beginning, we note that the hint given for the problem is a complete misdirection and is at no point useful. With that said, the first task is to recall how the annihilation operator is defined. If we remember that $a = uP + ivQ$ for some $u, v \in \mathbb{R}$ and $H = \hbar\omega(a^\dagger a + 1/2)$, we can multiply out $a^\dagger a$ and identify the coefficients u^2 and v^2 of P^2 and Q^2 with those in (2.3.1) to reverse engineer the definition

$$a = \frac{1}{\sqrt{2m\hbar\omega}} P + i\sqrt{\frac{m\omega}{2\hbar}} Q = \frac{1}{\sqrt{2m\hbar\omega}} \left(P + im\omega Q\right). \tag{2.4.1}$$

Now that we have the definition for a, we need to rewrite it in terms of $z = x + iy$ and $\bar{z} = x - iy$. The simplest way to do this is probably to invert these relations to find

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i} \tag{2.4.2}$$

and use them in (2.4.1). However, before we do so, we also need to determine the induced transformation in the momenta to $p_z, p_{\bar{z}}$. It is tempting to assume that the z-momenta satisfy the same algebraic relations as the z's themselves, but this is not correct.

There are two ways to deduce the correct relationship between the momenta. Using the chain rule to expand the x and y derivatives, we find

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\partial + \frac{\partial \bar{z}}{\partial x}\bar{\partial} = \partial + \bar{\partial}, \qquad \frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\partial + \frac{\partial \bar{z}}{\partial y}\bar{\partial} = i\partial - i\bar{\partial}, \tag{2.4.3}$$

which imply the relations

$$p_x = p_z + p_{\bar{z}}, \qquad p_y = i(p_z - p_{\bar{z}}) \tag{2.4.4}$$

Alternatively, we could observe that the differential is a linear operator, so $\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z})$, $\mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\bar{z})$. We then seek some expressions $\partial_x = a\partial + b\bar{\partial}$ and $\partial_y = c\partial + d\bar{\partial}$ for some $a, b, c, d \in \mathbb{C}$. We must satisfy the four relations $\mathrm{d}x(\partial_x) = 1, \mathrm{d}x(\partial_y) = 0, \mathrm{d}y(\partial_x) = 0$, and $\mathrm{d}y(\partial_y) = 1$. This system has a unique solution which is exactly (2.4.3), from which the same momenta relations follow.

In any case, we use the momenta and coordinate relations to rewrite the annihilation operator

$$\begin{aligned} a &= \frac{1}{\sqrt{2m\hbar\omega}} \left[\left(i(p_z - p_{\bar{z}}) - \frac{qB}{4c}(z + \bar{z}) \right) - i\frac{m\omega}{qB} \left(c(p_z + p_{\bar{z}}) + \frac{qB}{4i}(z - \bar{z}) \right) \right] \\ &= \frac{1}{\sqrt{2m\hbar\omega}} \left[\left(i(p_z - p_{\bar{z}}) - \frac{qB}{4c}(z + \bar{z}) \right) - i(p_z + p_{\bar{z}}) - \frac{qB}{4c}(z - \bar{z}) \right] \\ &= \frac{-1}{\sqrt{2m\hbar\omega}} \left[i2p_{\bar{z}} + \frac{qB}{2c} z \right]. \end{aligned} \tag{2.4.5}$$

Which is then the annihilation operator written in terms of $z, \bar{z}$.

Now, the problem also asks us to find the ground state wave function as a function of $z, \bar{z}$, $\psi_0(z, \bar{z})$. Since this is the ground state wave function, we know that it must be annihilated by the annihilation operator, $a\psi_0(z, \bar{z}) = 0$. Since $p_{\bar{z}} = -i\hbar\bar{\partial}$, it follows that ψ_0 must satisfy

$$\bar{\partial}\psi_0 = -\frac{qB}{4\hbar c}z. \tag{2.4.6}$$

This is now a separable differential equation which may be integrated readily to obtain

$$\ln \psi_0(z, \bar{z}) = -\frac{qB}{4\hbar c}z\bar{z} + w(z), \tag{2.4.7}$$

where $w(\bar{z})$ is an integration constant, which may depend only on z since we integrated with respect to $\bar{z}$. Exponentiating and defining $u(z) = e^{w(z)}$. we find

$$\psi_0(z, \bar{z}) = u(z, \bar{z}) \exp\left[-qBz\bar{z}/4\hbar c\right], \tag{2.4.8}$$

where we have written $u(z) = u(z, \bar{z})$ to match the problem statement, but obviously $\bar{\partial}u(z, \bar{z}) = 0$ since u does not actually depend on $\bar{z}$. This is the desired result.

As mentioned earlier, the hint for the problem is actually useless. In fact, the only use it might have is if you wished to prove that $\bar{\partial}u = 0$ is equivalent to analiticity, though the problem does not require this since the statement also tells you that $\bar{\partial}u = 0$.

3 Problem 2: Quantum Mechanics

3.1 Problem Statement (Rabbi Flopping, Solving ODE)

A spin-1/2 particle processes in a magnetic field $B_0\hat{\mathbf{z}}$ at the frequency $\omega_0 = \gamma B_0$. We turn on a small transverse radiofrequency field given by

$$\mathbf{B} = B_1 \cos(\omega t)\hat{\mathbf{x}} - B_1 \sin(\omega t)\hat{\mathbf{y}}, \tag{3.1.1}$$

so that the total field is $\mathbf{B} = B_1 \cos(\omega t)\hat{\mathbf{x}} - B_1 \sin(\omega t)\hat{\mathbf{y}} + B_0\hat{\mathbf{z}}$.

(a) Construct the 2×2 Hamiltonian matrix for this system.

(b) Let $\chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ be the two-component spinor at time t. Show that

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{i}{2}\left(\Omega e^{i\omega t}b + \omega_0 a\right), \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = \frac{i}{2}\left(\Omega e^{-i\omega t}a - \omega_0 b\right), \tag{3.1.2}$$

where $\Omega = \gamma B_1$.

(c) Now, simplify the equations by the substitutions $a(t) = A(t)e^{i\omega_0 t/2}$ and $b(t) = B(t)e^{-i\omega_0 t/2}$ to find the equations for $A(t)$ and $B(t)$. Solve these equations at the resonance by setting $\omega = \omega_0$. Decouple them by taking another derivative. Apply the initial condition $a(0) = 1$ and $b(0) = 0$ and sketch the probability of a transition to spin down, as a function of time. Can arbitrarily small B_1 flip the spin at resonance? Explain.

Hint: The spin-1/2 matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1.3}$$

3.2 Part (a)

For this first part, you simply had to know that $H = -\gamma \mathbf{B}\cdot\mathbf{S}$. However, if you did not know this, it would be sufficient to remember only that $H \propto \mathbf{B}\cdot\mathbf{S}$. The coefficient may then be reverse engineered to make the target expression match the equations we find. In any case, we are given the spin matrices, so we may explicitly write out the dot product as

$$\mathbf{B}\cdot\mathbf{S} = \frac{\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} B_0 & B_1 e^{i\omega t} \\ B_1 e^{-i\omega t} & -B_0 \end{pmatrix}. \tag{3.2.1}$$

Hence,

$$H = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B_1 e^{i\omega t} \\ B_1 e^{-i\omega t} & -B_0 \end{pmatrix}. \tag{3.2.2}$$

3.3 Part (b)

Now that we have the Hamiltonian, we need the equations of motion for the vector $\chi(t)$. Firstly, the problem statement calls this a spinor. In this instance, what the problem actually means is that χ is the finite-dimensional wave function, so it satisfies the Schrödinger equation. However, if we did not know this, we could always recall that the time-evolution operator is $e^{-iHt/\hbar}$. It then follows that $\chi(t) = e^{-iHt/\hbar}\chi(0)$. If we take a time derivative of this and identify $\chi(t)$ in the result, we do, in fact, find that exactly the Schrödinger equation determines the evolution of χ. So, we have

$$i\hbar\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B_1 e^{i\omega t} \\ B_1 e^{-i\omega t} & -B_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}. \tag{3.3.1}$$

If we perform the matrix multiplication on the RHS and identify $\omega_0 = \gamma B_0$ and $\Omega = \gamma B_1$, we find the desired result, (3.1.2).

3.4 Part (c)

Our first task for this part of the problem is to perform the indicated substitution, which yields

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t}e^{i\omega_0 t/2}+\frac{i\omega_0}{2}Ae^{i\omega_0 t/2}&=\frac{i}{2}\left(\Omega e^{i\omega_0 t/2}B+\omega_0 Ae^{i\omega_0 t/2}\right)\\ \frac{\mathrm{d}B}{\mathrm{d}t}e^{-i\omega_0 t/2}-\frac{i\omega_0}{2}Be^{-i\omega_0 t/2}&=\frac{i}{2}\left(\Omega e^{-i\omega_0 t/2}A-\omega_0 Be^{-\omega_0 t/2}\right),\end{aligned} \tag{3.4.1}$$

where we have already imposed the on-resonance condition, $\omega=\omega_0$. We notice that the last terms on the right and left hand sides cancel along with the exponentials, so we are left with the simplified system

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{i}{2}\Omega B,\qquad \frac{\mathrm{d}B}{\mathrm{d}t}=\frac{i}{2}\Omega A. \tag{3.4.2}$$

Now, at this point there are two ways to proceed. We could proceed as the question instructs and take and additional derivative of both equations in (3.4.2). That is,

$$\begin{aligned}\frac{\mathrm{d}^2A}{\mathrm{d}t^2}&=\frac{i}{2}\Omega\frac{\mathrm{d}B}{\mathrm{d}t}=-\frac{1}{4}\Omega^2A,\\ \frac{\mathrm{d}^2B}{\mathrm{d}t^2}&=\frac{i}{2}\Omega\frac{\mathrm{d}A}{\mathrm{d}t}=-\frac{1}{4}\Omega^2B.\end{aligned} \tag{3.4.3}$$

These are the equations of motion of a pair of harmonic oscillators, for which we know the solution already. However, when solving this system, we will note that by taking the extra derivative we have increased the order of the differential equations. This means the supplied initial conditions will be insufficient to determine the solution if we just solve (3.4.3). The trick of the matter is to take the general solution found for (3.4.3) and require that it also satisfy (3.4.2). This will eliminate some of the integration constants and allow the solution to be determined by the initial conditions supplied in the problem statement.

This procedure will work, but it takes a while and involves a fair bit of algebra to get the final solution. So, we will instead propose an alternate method of solution. To begin, we notice that the system (3.4.2) may be written in the matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}A\\B\end{pmatrix}=\frac{i}{2}\Omega\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}. \tag{3.4.4}$$

However, the matrix here is just the Pauli matrix σ_x. The formal solution to this equation is then

$$\begin{pmatrix}A\\B\end{pmatrix}=\exp\left[\frac{i}{2}\Omega t\sigma_x\right]\begin{pmatrix}A(0)\\B(0)\end{pmatrix}. \tag{3.4.5}$$

We may either recall that for a unit 3-vector $\hat{\mathbf{n}}$, $\exp(i\alpha\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}) = \cos(\alpha) + i\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\sin(\alpha)$ and specialize for our case $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, or if we do not recall this, note that $\sigma_x^2 = 1$ and expand the exponential in its Taylor series, recollecting the summation into odd and even powers of σ_x, then identifying the Taylor series for the sine and cosine. In either case, we find

$$\begin{pmatrix} A \\ B \end{pmatrix} = [\cos(\Omega t/2) + i\sigma_x \sin(\Omega t/2)] \begin{pmatrix} A(0) \\ B(0) \end{pmatrix}. \tag{3.4.6}$$

Now, we know the given initial conditions are $a(0) = 1$ and $b(0) = 0$. This implies the initial conditions $A(0) = 1$ and $B(0) = 0$ in our new variables. Hence, the motion is given by

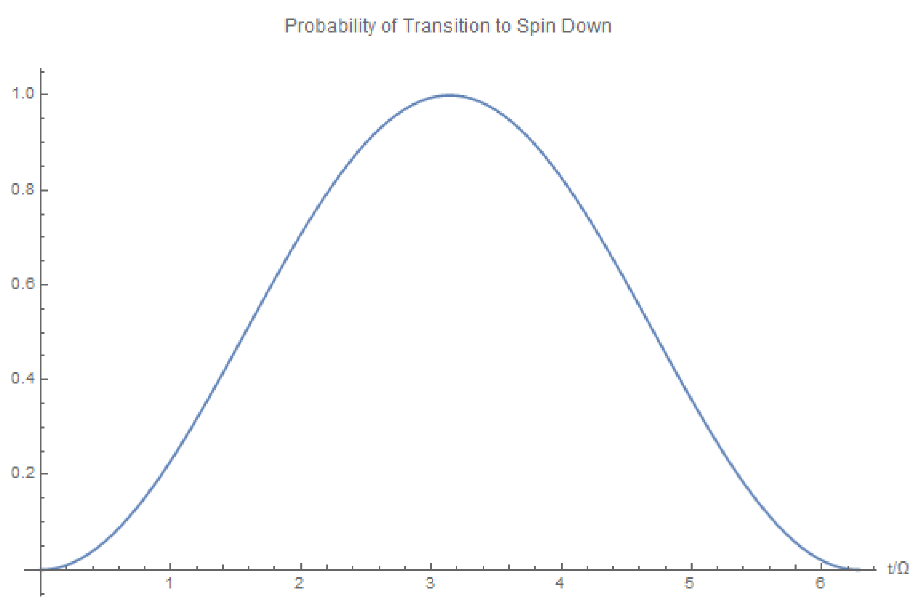
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \cos(\Omega t/2) \\ i\sin(\Omega t/2) \end{pmatrix}, \tag{3.4.7}$$

so the motion of the original variables is

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\Omega t/2)e^{i\omega_0 t/2} \\ i\sin(\Omega t/2)e^{-i\omega_0 t/2} \end{pmatrix}. \tag{3.4.8}$$

So, now that we have solved the equations of motion, the problem asks for the probability of a transition to spin down. So, if the current state is given by $\chi(t)$, we are interested in

$$P(|\chi\rangle \to |\downarrow\rangle, t) = |\langle\downarrow|\chi\rangle|^2 = \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \chi(t) \right|^2 = |i\sin(\Omega t/2)e^{-i\omega_0 t/2}|^2 = \sin^2(\Omega t/2). \tag{3.4.9}$$



Finally, the problem asks whether an arbitrarily small, but non-zero field strength B_1 is capable of causing the spin to flip on-resonance. This question is a bit vague, but it seems the answer is yes. Consider, $\Omega \propto B_1$, so if $B_1 \neq 0$, then $\omega \neq 0$. And, unless $\Omega = 0$, the transition probability (3.4.9) will be generically non-zero. As $\Omega \to 0$, the time it takes to reach the first maxima of the transition probability goes to infinity, but as long as we are willing to wait long enough, the probability of transition will become arbitrarily large.

4 Problem 3: Quantum Mechanics

4.1 Problem Statement (Commutator Algebras)

(a) Consider the lattice translation operator

$$T(a) = e^{-ia\hat{p}/\hbar} \tag{4.1.1}$$

where a is some constant and $\hat{p}$ is the momentum operator. Show that

$$T^\dagger(a)\hat{x}T(a) = \hat{x} + a. \tag{4.1.2}$$

(b) Show that $T(a)$ is unitary and show that $T(a)$ has eigenvalues of the form $e^{i\phi}$ where ϕ is real (you may assume that $\hat{p}$ is Hermitian).

(c) Consider the family of Hamiltonians which are periodic under shifts by a:

$$H = \frac{\hat{p}^2}{2m} + \sum_{n=-\infty}^{\infty} V(x - na). \tag{4.1.3}$$

You may assume that $V(x)$ goes exponentially fast to zero as $|x| \to \infty$ so the summation is convergent, and that $V(x)$ admits a power series expansion.

By examining the lattice translational symmetry of H or otherwise, prove that the Hamiltonian, H, has eigenstates $|E, k\rangle$ where k is a real parameter. These states must satisfy

$$H|E, k\rangle = E|E, k\rangle \tag{4.1.4}$$

and must be such that the wave function in position space defined by

$$u_k(x) = e^{-ikx}\langle x|E, k\rangle \tag{4.1.5}$$

are periodic functions with period a. That is,

$$u_k(x+a) = u_k(x). \tag{4.1.6}$$

What is the significance of the parameter k?

This is Block's theorem for periodic potentials (i.e. and energy eigenstate can be written as a Block wave times a periodic function).

4.2 Part (a)

There are several ways to tackle this first problem. We will not demonstrate them all here, but will pick a single method and simply mention the procedures for the others. Firstly, if you recall the Baker-Campbell-Hausdorff formula for this exact situation, the problem is finished in two lines. Failing that, we could use the result of part (b) (which does not depend on the result of part (a), so the logic would not be circular) to write $[\hat{x}, T(a)] = aT(a)$, which may be shown by expanding $T(a)$ as a power series and computing the commutators $[\hat{x}, \hat{p}^n] = i\hbar n\hat{p}^{n-1}$. This identity itself may be proved inductively or via the trick we employ to solve the problem directly.

We know that in position space, we are free to write $\hat{p} = -i\hbar\partial_x$. A similar expression exists for the position operator in momenta space, but with a positive sign. So, we are free to write $\hat{x} = i\hbar\partial_p$. We can always check the sign by putting this expression into the commutator $[\hat{x}, \hat{p}] = i\hbar$. Essentially, we are observing that $p, i\hbar\partial_p$ furnish a representation of the abstract Lie algebra, so the results we obtain with this representation will hold in any representation. In any case, noting that $\hat{x}$ acts on *all* objects to its right, we may write

$$e^{ia\hat{p}/\hbar} i\hbar \frac{\partial}{\partial p} e^{-ia\hat{p}/\hbar} = i\hbar e^{ia\hat{p}/\hbar}\left(-\frac{ia}{\hbar}e^{-ia\hat{p}/\hbar} + e^{-ia\hat{p}/\hbar}\frac{\partial}{\partial p}\right) = a + \hat{x} = \hat{x} + a, \tag{4.2.1}$$

as desired.

4.3 Part (b)

For unitarity, we require that $T^\dagger(a)T(a) = T(a)T^\dagger(a) = 1$. However, since we are on a lattice which is infinite in both directions (as opposed to a Hilbert's hotel type situation), the commutivity of $\hat{p}$ with itself suffices here.

To show that the eigenvalues of $T(a)$ have the form $e^{i\phi}$, we may proceed in one of two ways. Firstly, we way suppose that $|\lambda\rangle$ is an eigenvector of $T(a)$ so $T(a)|\lambda\rangle = \lambda|\lambda\rangle$. But then if we take these eigenvectors to be normalized,

$$1 = \langle\lambda|\rangle = \langle\lambda|T^\dagger(a)T(a)|\lambda\rangle = |\lambda|^2\langle\lambda|\lambda\rangle = |\lambda|^2, \tag{4.3.1}$$

so λ must have modulus one, and hence has the form $\lambda = e^{i\phi}$ for some $\phi \in \mathbb{R}$.

Alternatively, we could observe that if $|p\rangle$ is an eigenvector of $\hat{p}$, it must also be an eigenvector of $T(a)$ with eigenvalue $e^{-iap/\hbar}$, which again has the desired form. This is the preferred method since we will need to consider eigenvectors of $\hat{p}$ such that $\hat{p}|k\rangle = \hbar k|k\rangle$ for part (c).

4.4 Part (c)

Now, this part of the question is poorly worded as to rely on the reader already knowing what the answer is supposed to be. Regardless, everyone who takes the exam needs to figure out how to write something or other down. Unlike the wording might suggest on first reading, this problem is not actually instructing us to show that $[T(a), H] = 0$, it is only suggesting that we might want to do this.

So, we need to figure out what the quantum number k is supposed to be corresponding to in order to make the final result (4.1.6) possible. Well, the use of the letter k and the suggestion that we consider the translation operator suggests that k be some kind of momenta. With the benefit of hind-sight, we will in fact want to define $|k\rangle$ such that $\hat{p}|k\rangle = \hbar k|k\rangle$. So, to show that we may label the state vectors by E and k simultaneously, it will suffice to show that $T(a)$ and H commute since the $|k\rangle$ are also eigenvectors of $T(a)$.

So, since $T(a)$ is unitary, it will suffice to show that $T^\dagger(a)HT(a) = H$ to show that H and $T(a)$ commute. Since $T(a)$ clearly commutes with $\hat{p}^2$, we need only show that it commutes with the potential term of the Hamiltonian. Towards this end, we first compute $T^\dagger(a)\hat{x}^n T(a) = (\hat{x} + a)^n$ by inserting $1 = T(a)T^\dagger(a)$ between the factors of $\hat{x}$.

Now, since $V(x - na)$ admits a power series expansion, we may compute $T^\dagger(a)V(x - na)T(a)$ term by term to find $V(x - (n-1)a)$. But this now implies

$$T^\dagger(a)HT(a) = \frac{\hat{p}^2}{2m} + \sum_{n=-\infty}^{\infty} V(x - (n-1)a) = \frac{\hat{p}^2}{2m} + \sum_{n=-\infty}^{\infty} V(x - na) = H, \tag{4.4.1}$$

where we have reindexed the infinite sum to shift the value of n by one.

Now that we know $T(a)$ and H commute, we know that they are simultaneously diagonalizable and hence we are free to label our states by E and k, $|E, k\rangle$.

If we now take a look at (4.1.6) and begin to think how to prove this, we may write out the definition of the RHS, $u_k(x + a) = e^{-ikx}e^{-ika}\langle x + a|E, k\rangle$. Since we need this to become $u_k(x)$, we need some way of writing $\langle x + a|$ in terms of $\langle x|$. If the translation operator is named with any sense, it should be able to do this for us. So we compute

$$\hat{x}T(a)|x\rangle = T(a)T^\dagger(a)\hat{x}T(a)|x\rangle = T(a)(\hat{x} + a)|x\rangle = (x + a)T(a)|x\rangle, \tag{4.4.2}$$

so we conclude[4] $T(a)|x\rangle = |x+a\rangle$.

It now follows that

$$u_k(x+a) = e^{-ikx}e^{-ika}\langle x|T^\dagger(a)|E,k\rangle = e^{-ikx}e^{-ika}\langle x|e^{ika}|E,k\rangle = u_k(x), \tag{4.4.3}$$

as desired.

Finally, the problem asks us to interpret the quantity k. However, this is a bit vague, and it's not clear what the author of the problem wanted as an answer. More than likely, they wanted the student to respond that k is the lattice momenta since that's what condensed matter physicists like to talk about and this problem was clearly written by a condensed matter physicist.

5 Problem 4: Quantum Mechanics

5.1 Problem Statement (Time Independent Perturbation Theory)

Consider a two-level system with unperturbed energy levels such that

$$\mathbf{H}_0|1\rangle = \epsilon|1\rangle, \quad \mathbf{H}_0|2\rangle = -\epsilon|2\rangle. \tag{5.1.1}$$

Add a perturbation with off-diagonal elements only:

$$\mathbf{V} = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}. \tag{5.1.2}$$

(a) What are the exact eigenvalue of the total Hamiltonian, $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$?

(b) Assuming the perturbation is small, i.e. $V \ll \epsilon$, expand the exact energies to second order in V/ϵ.

(c) Show that this agrees with the results of second order non-degenerate perturbation theory.

(d) If $\epsilon \to 0$, the level are degenerate. How do the exact energy eigenvalues depend on V in this case?

(e) Show that for $\epsilon \neq 0$, $V \gg \epsilon$, the energy eigenvalues are linear in V.

[4]Technically, this equality is only true up to a phase, but since there are no superselection rules in this system, we are free to choose this phase to be identically equal to zero.

(f) In atoms and molecules, the field-free energy eigenstates are also eigenstates of parity. Show that the perturbation from applying an electric field, $\boldsymbol{\varepsilon}$, in the dipole approximation (i.e. $\mathbf{V} = -\mathbf{d} \cdot \varepsilon = -e\mathbf{r} \cdot \boldsymbol{\varepsilon}$) has only off-diagonal elements.

(g) In atoms, opposite parity states have nergy separations much bigger than the Stark energy shifts due to an electric field that can be applied in the laboratory. How does the Stark shift depend on $\boldsymbol{\varepsilon}$ in this case?

(h) In chemistry, you are typically taught that molecules have dipole moments and thus energy shifts linear in electric field (i.e. $\Delta E = -\mathbf{d} \cdot \boldsymbol{\varepsilon}$). Given that truly degenerate energy levels rarely (if ever) exist, how do you reconcile this?

5.2 Parts (a)-(e)

First, the total Hamiltonian is

$$\mathbf{H} = \begin{pmatrix} \epsilon & V \\ V^* & -\epsilon \end{pmatrix}. \tag{5.2.1}$$

The energy eigenvalues are then determined by $0 = \det(\mathbf{H} - \lambda) = -(\epsilon - \lambda)(\epsilon + \lambda) - |V|^2$ which implies $\lambda = \pm\sqrt{\epsilon^2 + |V|^2}$ are the exact energy eigenvalues.

To expand this in powers of V/ϵ, we write

$$\lambda = \pm\epsilon\sqrt{1 + \left|\frac{V}{\epsilon}\right|^2} \approx \pm\epsilon\left(1 + \frac{1}{2}\left|\frac{V}{\epsilon}\right|^2\right), \tag{5.2.2}$$

taking the first order expansion of $\sqrt{1+x}$, which is second order in V/ϵ.

Next we look at the the second order non-degenerate perturbation theory. First of all, since $\mathbf{V}$ has no on-diagonal terms, the first order perturbation, $E_n^{(1)} = \langle n|\mathbf{V}|n\rangle = 0$. For the second order perturbation, we recall the formula

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n|\mathbf{V}|m\rangle|^2}{E_n^{(0)} - E_m^{(0)}}. \tag{5.2.3}$$

We note that in this formula, the energy index we are currently looking at always comes first in the denominator. For the problem at hand, we have

$$\begin{aligned} E_1^{(2)} &= \frac{|\langle 1|\mathbf{V}|2\rangle|^2}{\epsilon + \epsilon} = \frac{1}{2}\frac{|V|^2}{\epsilon}, \\ E_2^{(2)} &= \frac{|\langle 1|\mathbf{V}|2\rangle|^2}{-\epsilon - \epsilon} = -\frac{1}{2}\frac{|V|^2}{\epsilon}, \end{aligned} \tag{5.2.4}$$

which both agree with the exact energy eigenvalues to second order in V/ϵ.

For part (d), we take $\epsilon \to 0$ in the exact formula to find $\lambda = \pm|V|$, which is then linear in V.

Finally, for part (e), we take $\epsilon \neq 0$, but $V \gg \epsilon$. Then we write

$$\lambda = \pm|V|\sqrt{1 + \left|\frac{\epsilon}{V}\right|^2} \approx= \pm|V|\left(1 + \frac{1}{2}\left|\frac{\epsilon}{V}\right|^2\right), \tag{5.2.5}$$

which, if we insist on keeping till only first order in ϵ/V, is linear in $|V|$, as requested.

5.3 Parts (f)-(h)

In part (f), we are asked to show that $\mathbf{V} = -e\mathbf{r}\cdot\boldsymbol{\varepsilon}$ has no on-diagonal elements knowing that the basis of eigenstates when $\boldsymbol{\varepsilon} = 0$ have definite parity. Let $\hat{\pi}$ be the parity operator and $|E_n\rangle$ be the $\boldsymbol{\varepsilon} = 0$ energy eigenstates. Then $\hat{\pi}|E_n\rangle = \eta_n|E_n\rangle$ where $\eta_n = \pm 1$. By definition, we know that $\hat{\pi}\mathbf{r}\hat{\pi} = -\mathbf{r}$. Then[5] $\hat{\pi}\mathbf{V}\hat{\pi} = -\mathbf{V}$. it follows that

$$\langle E_n|\mathbf{V}|E_n\rangle = \langle E_n|\hat{\pi}\hat{\pi}\mathbf{V}\hat{\pi}\hat{\pi}|E_n\rangle = -\eta_n^2\langle E_n|\mathbf{V}|E_n\rangle. \tag{5.3.1}$$

Hence, $\langle E_n|\mathbf{V}|E_n\rangle = -\langle E_n|\mathbf{V}|E_n\rangle$, which implies $\langle E_n|\mathbf{V}|E_n\rangle = 0$, so $\mathbf{V}$ has no on-diagonal elements.

For part (g), we note that the unperturbed levels are much further apart than the shifts. In the language of this problem, that would be the case $\epsilon \gg V$ so the energies are highly non-degenerate. As we showed in part (b), the energy shifts will be of order $\boldsymbol{\varepsilon}^2$. Alternatively, we could argue that because the system is highly non-degenerate, non-degenerate perturbation theory is good, but we already argued that on-diagonal elements of the Stark perturbation are zero by parity, so we know the first order correction will be zero, leaving the second order correction as the lowest order correction available.

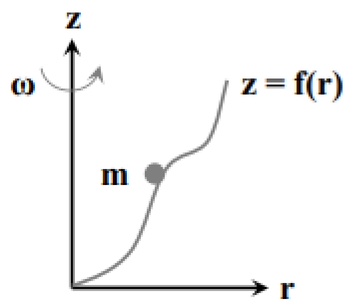
Finally, in part (h), we note that a linear dependence of the shift on the electric field comes from a degenerate, or nearly degenerate, perturbation of the states, as we saw in part (e). Suppose the conclusion to draw is that the energy levels in molecules are nearly degenerate. It is unclear what reconciliation is necessary since the case here and in part (g) are simply different limits. Perhaps it is worthwhile to point out that nearly-degenerate perturbations may have first-order dependence even though the perturbation is off-diagonal.

[5] There is an important point to be made here. The astute reader may have noticed that the electric field is supposed to transform under parity like a vector, and hence should pick up a sign. However, the electric field we have here is not dynamical, it is only a parameter in the problem. Another way of arguing this is to point out that $\boldsymbol{\varepsilon}$ is a c-number, not an operator, so $\hat{\pi}$ necessarily commutes with it. If we had a fully dynamical electric field as in QED, then we would need to account for the lack of commutivity.

6 Problem 5: Classical Mechanics

6.1 Problem Statement (Galilean Lagrangian Calculation)

Consider a point mass m sliding on a wire defined by the function $z = f(r)$, where $r = \sqrt{x^2 + y^2}$. The wire has a fixed shape and is rotating about the z-axis with an angular velocity ω. Consider the gravitational acceleration g (acting towards $-\hat{\mathbf{z}}$) and ignore any friction.



(a) Write down the Lagrangian $L(r, \dot{r}, t)$ for the mass.

(b) Using (a), find he equation of motion for $r(t)$. Then let r_0 be the (constant) radius of a fixed circular orbit. Derive the condition on $f(r)$ at $r = r_0$ for a circular orbit to exist. (Hint: show that $r'(r_0)/r_0$ equals a constant).

(c) Consider a small change in the circular orbit $r(t) = r_0 + \epsilon(t)$. What is the condition on $f(r)$ in order to have a stable circular orbit at $r = r_0$?

(d) From the Lagrangian, find the Hamiltonian $H(r, p, t)$, where p is the canonical momentum. Is the Hamiltonian conserved?

6.2 Part (a)

We know that for a discrete particles in Cartesian coordinates, $L = T - V$ where T is the Cartesian kinetic energy and V is the Cartesian potential energy. Rather than attempt this problem with Lagrange multipliers, it is more convenient to impose the constraints of the problem from the beginning. Since the problem has a cylindrical symmetry, we convert to cylindrical coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{6.2.1}$$

The constraints for the problem are $\theta = \omega t$ and $z = f(r)$, so the velocities may be computed to be

$$\dot{x} = \dot{r}\cos\omega t - \omega r \sin\omega t, \qquad \dot{y} = \dot{r}\sin\omega t + \omega r\cos\omega t, \qquad \dot{z} = \dot{r}f'(r). \tag{6.2.2}$$

Since $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, it follows that the kinetic energy written in cylindrical coordinates with constraints imposed must be

$$T = \frac{1}{2}m\left([1 + f'^{\,2}(r)]\dot{r}^2 + \omega^2 r^2\right). \tag{6.2.3}$$

Since the only force on the particle besides the constraints, which have already been imposed, is gravity, the potential energy is given by $V = mgz = mgf(r)$.

It now follows that the Lagrangian is given by

$$L = \frac{1}{2}m\left([1 + f'^{\,2}(r)]\dot{r}^2 + \omega^2 r^2\right) - mgf(r). \tag{6.2.4}$$

6.3 Part (b)

The Euler-Lagrange equations for a Lagrangian depending on only the first time derivative of the coordinate are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}. \tag{6.3.1}$$

We therefore compute

$$\frac{\partial L}{\partial r} = m\omega^2 r + mgf'(r), \qquad \frac{\partial L}{\partial \dot{r}} = m[1 + f'^{\,2}(r)]\dot{r}, \tag{6.3.2}$$

and hence the equations of motion are

$$m\ddot{r}[1 + f'^{\,2}(r)] + m\dot{r}^2 f'(r)f''(r) = m\omega^2 r - mgf'(r) \tag{6.3.3}$$

after computing

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = m(\ddot{r} + \ddot{r}f'^{\,2}(r) + 2\dot{r}^2 f'(r)f''(r)). \tag{6.3.4}$$

Now that we have the equations of motion, we are interested in finding a condition for having a stable orbit, so we suppose $r = r_0$, which is a constant. Then immediately all the derivatives of r vanish in (6.3.3) and we are left with

$$\frac{f'(r_0)}{r_0} = \frac{\omega^2}{g}, \tag{6.3.5}$$

which we take as the condition for the existence of a stationary orbit.

6.4 Part (c)

We now suppose that $r(t) = r_0 + \epsilon(t)$, and expand the equations of motion to first order in $\epsilon(t)$. immediately, the $\dot{r}^2$ term vanishes and we have

$$m\ddot{\epsilon}(1 + f'^{\,2}(r_0)) = m\omega^2(r_0 + \epsilon) - mg\left(f'(r_0) + \epsilon f''(r_0)\right), \tag{6.4.1}$$

where we have expanded $f'(r_0 + \epsilon) = f'(r_0) + \epsilon f''(r_0)$. Collecting terms now in derivatives of ϵ, we find

$$m\ddot{\epsilon}(1 + f'^{\,2}(r_0)) = -\epsilon\left(mgf''(r_0) - m\omega^2\right) - \left(mgf'(r_0) - m\omega^2 r_0\right). \tag{6.4.2}$$

But if we recall that $f'(r_0)/r_0 - \omega^2/g$, we see that the last term of (6.4.2) vanishes and we are left with equation of motion of a harmonic oscillator for ϵ.

To have a stable orbit then, we need this to be a stable harmonic oscillator. That is, we require $(mgf''(r_0) - m\omega^2) \geq 0$ for the "spring constant." This then implies the condition

$$f''(r_0) \geq \omega^2/g \tag{6.4.3}$$

to ensure that a stable orbit is possible.

6.5 Part (d)

We recall that the Hamiltonian is defined to be the Legendre transformation of the Lagrangian, $H = \dot{r}p - L$ where $p \equiv \frac{\partial L}{\partial \dot{r}}$. But we have already computed p in (6.3.2). It is then a matter of simple algebra to find

$$H = \frac{1}{2m}\frac{p^2}{1 + f'^{\,2}(r)} - \frac{1}{2}m\omega^2 r + mgf(r). \tag{6.5.1}$$

Finally, we are asked whether or not the Hamiltonian is conserved. It is because it does not depend explicitly on time and we know $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}$ since the Hamiltonian Poisson commutes with itself. We could have also seen that the Hamiltonian would not depend on time explicitly before computing it by noting $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, and the Lagrangian did not depend explicitly on time.

7 Problem 6: Classical Mechanics

7.1 Problem Statement (Formal Relativistic Hamiltonian Dynamics)

(In this problem, we will not consider radiation effects).

Consider a particle of charge q and mass m in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) Write the non-relativistic Hamiltonian for he particle and find and integrate he equations of motion.

(b) Write the relativistic energy-momentum relation for the particle. Guess or anyway write down the relativistic Hamiltonian for the particle in the uniform magnetic field, and find the equations of motion.

7.2 Part (a)

We recall that the Hamiltonian for a non-relativistic particle in an electromagnetic field is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi. \tag{7.2.1}$$

Since we have no electric field, it is convenient to choose the symmetric gauge, which is even more conveniently provided in the statement of problem 1 on this exam. The symmetric gauge has $\phi = 0$ and

$$\mathbf{A} = \frac{B}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}). \tag{7.2.2}$$

Hence, the Hamiltonian is given by

$$H = \frac{1}{2m}\left[(p_x + yBq/2)^2 + (p_y - xBq/2)^2 + p_z^2\right]. \tag{7.2.3}$$

The equations of motion are then given by

$$\begin{aligned} \dot{z} &= \tfrac{p_z}{m} \quad \dot{x} = \tfrac{1}{m}(p_x + yBq/2), \qquad \dot{y} = \tfrac{1}{m}(p_y - xBq/2) \\ \dot{p}_z &= 0, \quad \dot{p}_x = \tfrac{Bq}{2m}(p_y - xBq/2), \quad \dot{p}_y = -\tfrac{Bq}{2m}(p_x + yBq/2) \end{aligned} \tag{7.2.4}$$

We may immediately integrate the z-equations to find $z(t) = z_0 + v_z t$, so we need only worry about the x and y equations.

Now at this point we are faced with a bit of an issue because there is not an obvious symmetry to the system (7.2.4) which is easily exploitable. There are several strategies we could pursue to solve this system. The first would be to find a set of conserved quantities and gain some algebraic equations to replace the differential equations. For the sake of an exam, this is really only a possibility if we already know what quantities should be conserved, which typically means we already know what the solution is supposed to be – in this case we know we are dealing with a cyclotron motion, so the radius of the motion in the xy-plane will be conserved. But showing that this is, in fact, a conserved quantity and then computing the reduced system of equations will take quite a while in the scheme of a timed exam.

The second strategy we could pursue would be to come up with a canonical transformation which sends the Hamiltonian to a known form, such as the form of a harmonic oscillator. We may notice that the quantities within the squares of (7.2.3) taken to be the new P and Q could be good candidates, and in fact we know that these will work up to constant prefactors because we applied exactly this transformation in problem 1 on this exam. The issue with this, however, is the mapping $(p_x, p_y, x, y) \mapsto (P, Q)$ is obviously not invertable, so while we can find the solution for P and Q without much difficulty, we would not be able to uniquely determine the solution in the original coordinates. We would need to find another conjugate pair (P', Q') such that the entire transformation $(p_x, p_y, x, y) \mapsto (P, P', Q, Q')$ is canonical. This would require us to compute the partial generating function for the known part of the transformation and then deduce the form of the remaining portion of the transformation.

The final strategy, which we will actually employ here, will be to instead solve the Euler-Lagrange equations since the reader has almost certainly solve for the motion of a cyclotron from the Lorentz force law. This requires us to effectively go from the Hamiltonian formulation to the Lagrangian formulation of the problem, though as we will see it will not be necessary to actually write down the Lagrangian for the system. Typically, we start with a Lagrangian and convert to a Hamiltonian description, for which we know to compute $p = \frac{\partial L}{\partial \dot{q}}$ which then gives $\dot{q} = \dot{q}(p, q)$. To do the reverse, we actually use the relation $\dot{q} = \frac{\partial H}{\partial p}$, which then gives us $p = p(\dot{q}, q)$. Using this relation in the $\dot{p}$ equation gives the reverse correspondence between the Euler-Lagrange equations and Hamilton's equations.

For the system at hand, we find

$$p_x = m\dot{x} - \frac{Bq}{2} y, \qquad p_y = m\dot{y} + \frac{Bq}{2} x. \tag{7.2.5}$$

Using these relations, we may now write the $\dot{p}$ equations, after some simplification, in the form

$$\ddot{x} = \frac{qB}{m}\dot{y}, \qquad \ddot{y} = -\frac{qB}{m}\dot{x}, \tag{7.2.6}$$

which should be the familiar equations of motion for the cyclotron. There are three ways to solve this system. The first is to take an extra time derivative to decouple the equations. But I am really not a fan of this method for solving differential equations. A better way would be to define the complex variable $z = \dot{x} + i\dot{y}$, in terms of which the system above may be written $\dot{z} = -i\frac{qB}{m} z$. From here, the solution is clearly a complex exponential and the general solution follows readily.

There is, however, a third method which is optimal for flexing. We begin by observing that if $a = \dot{x}$ and $b = \dot{y}$, we may write (7.2.6) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{qB}{m}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\frac{qB}{m}\sigma_y \begin{pmatrix} a \\ b \end{pmatrix} \tag{7.2.7}$$

where σ_y is the Pauli y. The solution is then given by the matrix exponential of the Pauli matrix, which we know[6]. Then

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \exp\left(i\frac{qB}{m}\sigma_y t\right)\begin{pmatrix} a(0) \\ b(0) \end{pmatrix}, \tag{7.2.8}$$

which we integrate to find x and y. Integrating this, we note that σ_y is its own inverse, so

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - i\frac{m}{qB}\sigma_y \exp\left(i\frac{qB}{m}\sigma_y t\right)\begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix}, \tag{7.2.9}$$

which we may write explicitly as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \frac{1}{\omega}\begin{pmatrix} \sin(\omega t) & -\cos(\omega t) \\ \cos(\omega t) & \sin(\omega t) \end{pmatrix}\begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} \tag{7.2.10}$$

which is the usual thing[7]. For additional flex, note that the matrix, which is σ_y times the matrix exponential, is a unitary operator, and hence $(x(t) - x_0)^2 + (y(t) - y_0)^2$ is clearly a constant, giving us circular motion as expected.

7.3 Part (b)

We will begin from the Lagrangian, $L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{v}\cdot\mathbf{A}$. It follows that

$$\mathbf{p} \equiv \frac{\partial L}{\partial \mathbf{v}} = m\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} + q\mathbf{A}. \tag{7.3.1}$$

Rewriting this as

$$m\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \mathbf{p} - q\mathbf{A}, \tag{7.3.2}$$

we may square it and solve for v^2 to find

$$v^2 = c^2\frac{(\mathbf{p} - q\mathbf{A})^2}{m^2c^2 + (\mathbf{p} - q\mathbf{A})^2}, \tag{7.3.3}$$

from which we compute

$$1 - \frac{v^2}{c^2} = \frac{m^2c^2}{m^2c^2 + \mathbf{p} - q\mathbf{A})^2}, \qquad \mathbf{v} = c\sqrt{\frac{1}{m^2c^2 + (\mathbf{p} - q\mathbf{A})^2}}(\mathbf{p} - q\mathbf{A}). \tag{7.3.4}$$

[6] See the solution for problem 2 on this exam.

[7] We note that (x_0, y_0) is not the initial condition for x and y, but rather are just some integration constants which we will find to be the center of our cyclotron.

Hence, we may rewrite the Lagrangian in terms of the momenta by

$$L = -m^2c^2\chi + q\chi(\mathbf{p}\cdot\mathbf{A} - q\mathbf{A}^2), \tag{7.3.5}$$

where we have defined $\chi = c\sqrt{\frac{1}{m^2c^2+(\mathbf{p}-q\mathbf{A})^2}}$ for brevity. Next, since

$$\mathbf{v}\cdot\mathbf{p} = \chi(\mathbf{p}^2 - q\mathbf{A}\cdot\mathbf{p}), \tag{7.3.6}$$

it follows that the Hamiltonian is given by

$$\begin{aligned} H &= \mathbf{v}\cdot p - L = \chi\left[\mathbf{p}^2 - 2q\mathbf{A}\cdot\mathbf{p} + q^2\mathbf{A}^2 + m^2c^2\right] = \chi\left[(\mathbf{p}-q\mathbf{A})^2 + m^2c^2\right] \\ &= c\sqrt{(\mathbf{p}-q\mathbf{A})^2 + m^2c^2}. \end{aligned} \tag{7.3.7}$$

It then follows that the equations of motion are given by

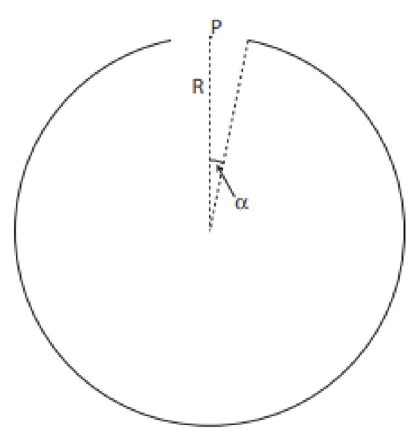
$$\dot{\mathbf{r}} = \frac{c^2}{H}(\mathbf{p} - q\mathbf{A}), \qquad \dot{p}_i = q\frac{c^2}{H}(p_j - qA_j)\partial_i A^j, \tag{7.3.8}$$

with summation over the 3-index j implied.

8 Problem 7: Electromagnetism

8.1 Problem Statement (Electrostatic Calculation)

A spherical shell of radius R is uniformly charged so that the charge per unit area on the surface is σ. You take a sword and chop off the very top of the sphere, so that there is a hole at the apex with polar opening angle α, as shown below.



(a) In the limit that the angle α is small (so that the diameter of the opening is much smaller than the radius of the sphere), calculate the electric field at the center of the sphere (magnitude and direction).

(b) Assuming the same limit, calculate the electric field at point P in the diagram (in the opening, at the location where the apex of the sphere used to be before I sliced off the top).

8.2 Part (a)

Since it is not clear from the problem statement, we will also assume that the aperture is intended to be circular and that the surface charge density does not change when the section of sphere was removed.

So, here's the approximation we will be making. Since the solid angle removed was small, we will approximate it by a disk of radius $R\sin\alpha \approx R\alpha$. Then, since Maxwell's equations are linear, the solution of this problem is equivalent to the sum of the solution for a completed sphere and the solution for a disk located at the removed section with surface charge density $-\sigma$ so the total density at the hole is zero.

Now, by Gauss' law, we know that the field inside the completed sphere is zero, while the field outside the sphere is $\mathbf{E} = \frac{4\pi R^2\sigma}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\rho}}}{r^2} = \frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\hat{\boldsymbol{\rho}}$. So, we just need the field due to the disk.

To find the field due to a disk of radius R, let's consider a disk of surface charge density σ located at the origin, the plane of the disk orthogonal to the z-axis. The field is then

$$\begin{aligned}\mathbf{E}(z\hat{\mathbf{z}}) &= \int_{\text{Disk}} \frac{\sigma \mathrm{d}A'}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \int_0^R \frac{2\pi\sigma r'\mathrm{d}r'}{4\pi\epsilon_0}\frac{z\hat{\mathbf{z}}}{(z^2+r'^2)^{3/2}} \\ &= \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R \mathrm{d}r'\frac{r'}{(z^2+r'^2)^{3/2}} = \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\frac{|z|-\sqrt{z^2+R^2}}{|z|\sqrt{z^2+R^2}},\end{aligned} \tag{8.2.1}$$

where we have used the symmetry of the system to argue that only the z-component of the electric field survives the integration on axis.

Now for the system at hand, the surface charge density of the disk is $-\sigma$, the radius is αR, and we must be careful that the z in (8.2.1) is the distance from the center of the disk, not the center of our sphere, and hence we must take $d = z - R$ to be the distance from the disk. Thus, the field due to the disk in the problem at hand is

$$\mathbf{E}(z\hat{\mathbf{z}}) = -\frac{\sigma d\hat{\mathbf{z}}}{2\epsilon_0}\frac{|d|-\sqrt{d^2+\alpha^2R^2}}{|d|\sqrt{d^2+\alpha^2R^2}} = -\frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{d}{|d|}\frac{|d|-\sqrt{d^2+\alpha^2R^2}}{\sqrt{d^2+\alpha^2R^2}}. \tag{8.2.2}$$

So to find the field at the center of the sphere, we must add the field due to the disk with the field due to the completed sphere. However, we already argued that the field at the center due to the completed sphere must be zero, so we are just left with the field due to the disk, which we evaluate at $z = 0$, which is $d = R$. The field is then

$$\mathbf{E}(0) = -\frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{1-\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}} \approx -\frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[1-\left(1-\frac{1}{2}\alpha^2\right)\right] = \frac{\sigma\alpha^2}{4\epsilon_0}\hat{\mathbf{z}}. \tag{8.2.3}$$

8.3 Part (b)

Since we already know the field (8.2.2), we just need to evaluate it at $z = R$, or $d = 0$. There is, however, a nicky point to watch out for when we do this. We know that since in the actual situation of interest, there is no surface charge at the location P, so the field should be continuous when crossing P. However, both the disk and the completed sphere have surface charge densities at the location P, so their fields are not continuous across P, but their discontinuities must cancel exactly so the sum is continuous again.

It is then simpler to evaluate the field as we come to the point P from the center of the sphere, so we need only to evaluate the field due to the disk. Doing so, we find

$$\mathbf{E}(P) = \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}. \tag{8.3.1}$$

While this would be good enough for the exam, for the sake of this document, we will also check that the field is correct when we evaluate the limit from the outside of the sphere. Doing so, we find

$$\mathbf{E}(P) = \frac{\sigma}{\epsilon_0}\frac{R^2}{R^2}\hat{\mathbf{z}} - \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0} = \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}. \tag{8.3.2}$$

Nice.

9 Problem 8: Electromagnetism

9.1 Problem Statement (Electromagnetic Angular Momentum)

A solenoid of radius R with n turns per unit length carries a stationary current I. Two hollow cylinders of length L are fixed coaxially and are free to rotate. One cylinder of radius a is inside the coil ($a < R$) and carries the uniformly distributed charge Q. The outer cylinder of radius b ($b > R$) carries the charge $-Q$. If the current is switched off, the cylinders start to rotate.

(a) Calculate the angular momentum of each cylinder.

(b) Calculate the total angular momentum at the end and explain where it is coming from.

9.2 Part (a)

Though the problem does not say so, we neglect edge effects in the fields. We do this because if we do not, it is not possible to solve the problem by hand. With this, however, we find by Gauss' and Amperé's laws the fields

$$\mathbf{E} = \begin{cases} 0, & r < a. \\ \frac{Q\hat{\boldsymbol{\rho}}}{2\pi\epsilon_0 Lr}, & a < r < b. \\ 0, & b < r. \end{cases} \qquad \mathbf{B} = \begin{cases} \mu_0 nI\hat{\mathbf{z}}, & r < R. \\ 0, & R < r. \end{cases} \tag{9.2.1}$$

where we have taken the current to flow in the $\hat{\boldsymbol{\theta}}$ direction.

Now, we need to find the angular momentum of the two cylinders. Well, we know that the time derivative of the angular momentum is the torque, and the torque is given by $\mathbf{r} \times \mathbf{f}$ where $\mathbf{f}$ is the Lorentz force,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \int_\Sigma \mathrm{d}A\mathbf{r} \times \mathbf{f}, \tag{9.2.2}$$

where $\mathbf{r}$ points from the origin to a point on one of the cylinders. Since the axes and radii of the cylinders are fixed, the L_θ and L_ρ components of $\mathbf{L}$ are constrained to be zero. This means that the z-component of $\mathbf{r}$ in (9.2.2) may be neglected since when crossed with any other vector, it will not produce a vector again in the z-direction. Hence, we only need to compute

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = \int_\Sigma \mathrm{d}A(rf_\theta - \theta f_\rho), \tag{9.2.3}$$

but the latter term in the integrand integrates to zero since the system is cylindrically symmetrical[8]. Thus, we need only write

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = 2\pi LR_c^2 f_\theta(R_c), \tag{9.2.4}$$

where R_c equals a or b, depending on the cylinder we are looking at.

[8] This is a bit of a hand-wavy argument. To make it precise, we would really need to distinguish explicitly between the coordinates and the vectors of the tangent space at a point since the two are not quite the same in cylindrical coordinates. Instead, we rely on the reader's intuition that a force acting radially which does not vary azimuthally will not torque the cylinder about the z-axis.

So, let's look at the Lorentz force to compute f_θ. Since $\mathbf{f} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, we note immediately that the static electric field (9.2.1) does not contribute to f_θ. Furthermore, we note that $\mathbf{v} \times \mathbf{B} \propto \hat{\boldsymbol{\theta}} \times \hat{\mathbf{z}} \propto \hat{\boldsymbol{\rho}}$, so the magnetic field (9.2.1) also fails to contribute to the system's torque.

However, as the current is turning off, the static magnetic field in (9.2.1) changes, which induces an electric field by Faraday's law[9]. To compute this electric field, we first compute the magnetic flux through a disk of radius r whose normal is aligned with the z-axis,

$$\Phi_B(r) = 2\pi \int_0^r r'\mathrm{d}r' B_z = \pi\mu_0 n I \min\{r, R\}^2, \tag{9.2.5}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_B(r) = \pi\mu_0 n \min\{r, R\}^2 \frac{\mathrm{d}I}{\mathrm{d}t}. \tag{9.2.6}$$

The electric field induced by this changing flux satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_B(r) = -\oint \mathrm{d}\boldsymbol{\ell} \cdot \mathbf{E} = -2\pi r E_\theta, \tag{9.2.7}$$

hence

$$E_\theta = -\frac{1}{2}\mu_0 n \frac{\min\{r, R\}^2}{r}\frac{\mathrm{d}I}{\mathrm{d}t}. \tag{9.2.8}$$

It now follows that time derivative of the angular momentum of the inner cylinder is

$$\frac{\mathrm{d}L_{a,z}}{\mathrm{d}t} = -\frac{1}{2}\mu_0 n Q a^2 \frac{\mathrm{d}I}{\mathrm{d}t}, \tag{9.2.9}$$

which which we deduce

$$L_{a,z} = \frac{1}{2}\mu_0 n Q I a^2 \tag{9.2.10}$$

since $\Delta I = -I$ and $\Delta L_{a,z} = L_{a,z}$.

In much the same fashion, just being careful about the computation of $\min\{b, R\}$ and that the charge should be $-Q$, we compute the angular momentum of the outer cylinder to be

$$L_{b,z} = -\frac{1}{2}\mu_0 n Q I R^2. \tag{9.2.11}$$

[9] Now, spoilers, this induced electric field is going to generate a torque. But we might also think that, as this torque is applied and the cylinders begin to spin, there will be a current which creates a magnetic field. However, since the induced currents will again be in the $\hat{\boldsymbol{\theta}}$ direction, the magnetic field will be along the $\hat{\mathbf{z}}$ just like another solenoid, so it produces no torque and we are safe to ignore it.

9.3 Part (b)

Now, the answer to the question "where did the momentum come from" is, of course, the field momentum, with the total final momentum in the cylinders being

$$L_{a,z} + L_{b,z} = \frac{1}{2}\mu_0 n Q I (a^2 - R^2). \tag{9.3.1}$$

We may, in fact, compute the initial angular momentum to be

$$\mathbf{L}_{EM} = \frac{1}{c^2}\int \mathrm{d}V \mathbf{r} \times \mathbf{S} = -\frac{2\pi L}{\mu_0 c^2}\hat{\mathbf{z}}\int_a^R r\mathrm{d}r \frac{\mu_0 n I Q}{2\pi\epsilon_0 L} = \frac{1}{2}\mu_0 n Q I (a^2 - R^2)\hat{\mathbf{z}} \tag{9.3.2}$$

It is interesting that this is equal to the final angular momentum of the cylinders because conservation of angular momentum would then imply that there is zero field momentum after the motion has completed. However, after the process has completed, we have two charged cylinders rotating. This means that there will be currents circulating in the $\hat{\boldsymbol{\theta}}$ direction – effectively creating two solenoids. So, we have the electric field which is produced by the existence of the charges, and also the magnetic field between the two now-solenoids. These fields will but just as orthogonal as the original fields (9.2.1), though their magnitudes and regions of non-zero magnitude will be different. This means the Poynting vector will again be non-zero, and will have a component again in the $\hat{\boldsymbol{\theta}}$ direction, so the field momenta after the process has concluded should still be non-zero. It is unclear to me what the source of this apparent non-conservation is, though there are three likely candidates. It could be due to our neglect of the fringe fields, the approximate iterative solution to Maxwell's equations, or the neglect of the external supply governing the original current I. Whatever the case may be though, it is quite surprising that the issue works out to cancel the final field momenta *exactly*.

10 Problem 9: Electromagnetism

10.1 Problem statement (Dielectric Scattering)

Calculate the scattering cross section for unpolarized light of a small dielectric sphere with electric susceptibility χ and radius $a \ll \lambda$.

(a) First calculate the induced dipole moment of a sphere in an external field.

(b) Then use this induced dipole to calculate the radiated power and then the cross section.

(c) What changes for $a \sim \lambda$? Do not calculate the cross section, just qualitatively explain the difference with respect to the main case of this problem and what approach you would follow to calculate this.

(d) Can you estimate the cross section in the opposite limit, $a \gg \lambda$?

10.2 Part (a)

Since the wavelength of the of the incident wave is much longer than the diameter of the sphere, we are free to approximate the field at the sphere as spatially constant. We call $\mathbf{E}_i = E_0 \hat{\mathbf{z}} e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega t)}$ the incident wave. We also note that if $a \ll \lambda$, then the frequency is small, so we are also free to approximate the solution by its first order classical perturbation – the solution to the time-independent problem evaluated at the time-dependent field. So, it will be sufficient to solve the potential theory problem to find the induced electric field.

To this end, we exploit the azimuthal symmetry of the problem to expand the potential in the interior and exterior regions in terms of the Legendre polynomials,

$$\phi_{in} = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta), \qquad \phi_{out} = \phi_{ext} + \sum_{k=0}^{\infty} \frac{B_k}{r^{k+1}} P_k(\cos\theta). \tag{10.2.1}$$

To determine the coefficients A_k and B_k, we must require that the potentials match at the boundary of the sphere and that the discontinuity in the normal component of the displacement field is equal to the free surface charge density, of which there is none. To simplify the calculation, however, we will rely on past experience and simply recall from the get go that $A_k = B_k = 0$ for $k > 1$. Furthermore, we point out that the potential for the external field is given by $\phi_{ext} = -E_i z = -E_i r\cos\theta = -E_i r P_1(\cos\theta)$. Hence, the potentials we are left with are

$$\phi_{in} = A_0 + A_1 r P_1(\cos\theta), \qquad \phi_{out} = \frac{B_0}{r} + \left(\frac{B_1}{r^2} - E_i r\right) P_1(\cos\theta). \tag{10.2.2}$$

Imposing the constraint at the boundary for the displacement field is equivalent to forcing

$$\epsilon \frac{\partial \phi_{in}}{\partial r}\bigg|_a = \epsilon_0 \frac{\partial \phi_{out}}{\partial r}\bigg|_a \tag{10.2.3}$$

where $\epsilon = (1+\chi)\epsilon_0$. In terms of the coefficients to be determined, this requires

$$\epsilon A_1 P_1(\cos\theta) = -\epsilon_0 \frac{B_0}{a^2} - \epsilon_0 \left(2\frac{B_1}{a^3} + E_i\right) P_1(\cos\theta). \tag{10.2.4}$$

Since we require this to hold across the entire sphere, the above condition must hold for all θ, and hence the orthogonality of the Legendre polynomials forces $B_0 = 0$ and

$$A_1 = -\frac{\epsilon_0}{\epsilon}\left(2\frac{B_1}{a^3} + E_i\right). \tag{10.2.5}$$

So now, the continuity condition on the scalar potential implies

$$A_0 - a\frac{\epsilon_0}{\epsilon}\left(2\frac{B_1}{a^3} + E_i\right) P_1(\cos\theta) = \left(\frac{B_1}{a^2} - aE_i\right) P_1(\cos\theta), \tag{10.2.6}$$

which must again hold for all θ, so $A_0 = 0$ and

$$-a\frac{\epsilon_0}{\epsilon}\left(2\frac{B_1}{a^3} + E_i\right) = \left(\frac{B_1}{a^2} - aE_i\right). \tag{10.2.7}$$

Solving this for B_1 and putting the resulting expressions back in terms of χ, we find

$$\phi_{in} = -\frac{3}{3+\chi}E_i r P_1(\cos\theta), \qquad \phi_{out} = E_i\left(\frac{3}{3+\chi}\frac{a^3}{r^2} - r\right) P_1(\cos\theta). \tag{10.2.8}$$

Now, we note that $rP_1(\cos\theta) = r\cos\theta = z$. It then follows that $\phi_{in} = -\frac{3}{3+\chi}E_i z$, and hence the electric field interior to the sphere is $\mathbf{E}_{in} = \frac{3}{3+\chi}E_i\hat{\mathbf{z}}$. We know by definition that $\mathbf{P} = \epsilon_0\chi\mathbf{E} = \frac{3\epsilon_0\chi}{3+\chi}E_i\hat{\mathbf{z}}$. Since the integral of $\mathbf{P}$ is the total dipole moment, it follows that

$$\mathbf{p} = \int \mathrm{d}V\mathbf{P} = \frac{4}{3}\pi a^3\frac{3\epsilon_0\chi}{3+\chi}E_i\hat{\mathbf{z}} = \pi\epsilon_0\frac{4\chi}{3+\chi}a^3 E_i\hat{\mathbf{z}} = \frac{4\pi\chi\epsilon_0}{3+\chi}E_0 a^3 e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega t)}. \tag{10.2.9}$$

We note that we are only justified in writing $\mathbf{E}_{in} = \frac{3}{3+\chi}E_i\hat{\mathbf{z}}$ because the a is small compared to the distance light travels per cycle, and so the retarded time is negligibly different from the frame-time. If this were not the case, then we would need to use the chain rule in computing the gradient. This is why we don't bother calculating the electric field outside the sphere here – the calculation would be at odds with the approximations we have made thus far.

10.3 Part (b)

Now that we have the induced dipole moment, we can compute the radiated power of the dipole and the scattering cross-section of the process. To calculate the power, we can just recall the formula for the power radiated by a dipole,

$$\frac{\mathrm{d}P_{dipole}}{\mathrm{d}\Omega} = \frac{\mu_0}{16\pi^2 c}|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{ret}|^2. \tag{10.3.1}$$

This is mostly a formula to be remembered because dipole radiation is the most common thing we can be asked to compute. However, we can still think about how we should go about remembering this formula, and the numeric prefactor in particular.

To this end, recall that the Poynting vector, $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ is the energy flux – that is, the energy per second per area. Let us consider the power passing through the surface element $r^2\mathrm{d}\Omega$. The power differential must then be $\mathrm{d}P = \hat{\mathbf{r}}\cdot\mathbf{S}r^2\mathrm{d}\Omega$, where we have assumed that the surface element is oriented to be on a sphere of radius r so that $\hat{\mathbf{r}}$ points to the surface element of interest. It should be noted that in older terminologies, such as Landau, this quantity $\mathrm{d}P$ is known as the intensity, $\mathrm{d}I$.

Next, we note that

$$\int \mathrm{d}V'\mathbf{j}(\mathbf{r}',t) = \sum q_k\mathbf{v}_k = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}V'\mathbf{r}'\rho = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{10.3.2}$$

now, as was pointed out in the disclaimer for this document, this is not a rigorous justification, but more of a mnemonic for remembering that the integral over all space of the current density should be equal to the first time derivative of the electric dipole moment[10].

Now, let us recall that the Green function for the vector potential is given by[11]

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}V'\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}. \tag{10.3.3}$$

Since we are looking to talk about radiation, we are interested in the far-field approximation in which

$$|\mathbf{r}-\mathbf{r}'| \approx r\left(1-\frac{r'}{r}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right). \tag{10.3.4}$$

With this, we are free to write the far field, or radiation, vector potential in the form

$$\mathbf{A}_{rad}(\mathbf{r},t) = \frac{\mu_0}{4\pi r}\int \mathrm{d}V'\mathbf{j}(\mathbf{r}',t-r/c+\hat{\mathbf{r}}\cdot\mathbf{r}'/c), \tag{10.3.5}$$

and by nearly identical arguments,

$$\phi_{rad}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}V'\rho(\mathbf{r}',t-r/c+\hat{\mathbf{r}}\cdot\mathbf{r}'/c). \tag{10.3.6}$$

[10] It is, however, worth pointing out that this is only true for systems of finite physical extent so the current density goes to zero sufficiently fast at infinity.

[11] We note that this is Lorenz gauge.

In fact, since the retarded time comes up so often, we define the symbol $t^*(\mathbf{r}, \mathbf{r}', t) = t - |\mathbf{r} - \mathbf{r}'|/c \approx t - r/c + \hat{\mathbf{r}} \cdot \mathbf{r}'$ for the retarded time. In this notation, we have

$$\begin{aligned} \mathbf{A}_{rad}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi r} \int \mathrm{d}V' \mathbf{j}(\mathbf{r}', t^*(\mathbf{r}, \mathbf{r}', t)), \\ \phi_{rad}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0 r} \int \mathrm{d}V' \rho(\mathbf{r}', t^*(\mathbf{r}, \mathbf{r}', t)). \end{aligned} \tag{10.3.7}$$

In terms of these fields, we could compute the electric and magnetic radiation fields. We demonstrate how to get the magnetic field and then will just state the result for the electric field. To perform this computation, we require $\nabla \times \mathbf{A}_{rad}$. This could be a bit of a mess, but remember that we are in the far field approximation, so we are only keeping terms to order $1/r$ in the fields[12]. As such, $\nabla(1/r) \approx 0$ for us. Hence, the curl commutes into the integral in (**??**) and we convert the derivative into a time derivative via

$$\frac{\partial}{\partial x^a} j_b(\mathbf{r}', t - r/c + \hat{\mathbf{r}} \cdot \mathbf{r}') \approx -\frac{\hat{r}_a}{c} \frac{\partial}{\partial t} j_b(\mathbf{r}', t - r/c + \hat{\mathbf{r}} \cdot \mathbf{r}'). \tag{10.3.8}$$

Hence,

$$\mathbf{B}_{rad}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}}}{r} \times \int \mathrm{d}V' \frac{\partial}{\partial t} \mathbf{j}(\mathbf{r}', t - r/c + \hat{\mathbf{r}} \cdot \mathbf{r}') = -\frac{\hat{\mathbf{r}}}{c} \times \frac{\partial}{\partial t} \mathbf{A}_{rad}(\mathbf{r}, t). \tag{10.3.9}$$

The computation for the electric field is essentially the same, but requires us to use the continuity equation to convert the time derivative of the charge density to a derivative of the current density. The result is

$$\mathbf{E}_{rad} = -\hat{\mathbf{r}} \times c\mathbf{B}_{rad}. \tag{10.3.10}$$

This should be more or less reminiscent to the quite general free-field result, $|\mathbf{E}| = c|\mathbf{B}|$. in particular, we note that these relations imply that $\mathbf{S} \propto \hat{\mathbf{r}}$.

So, we've done all of this, how do we make use of any of it? Our goal has been to compute the power radiated to infinity, which means the computation of $\mathrm{d}P$, which itself means the computation of $\hat{\mathbf{r}} \cdot \mathbf{S}$. However, as we have already noted the directionality of $\mathbf{S}$, we find that we need only compute $\mathrm{d}P = r^2 |\mathbf{S}| \mathrm{d}\Omega$. This is fortunate since

$$|\mathbf{S}| = \frac{1}{\mu_0} |\mathbf{E}_{rad} \times \mathbf{B}_{rad}| = \frac{1}{\mu_0 c} |\mathbf{E}_{rad}|^2 = \frac{c}{\mu_0} |\mathbf{B}_{rad}|^2. \tag{10.3.11}$$

[12] As another way to argue for this approximation, note that if we has a term of order $n > 1$ in $1/r$, then the definition given above for power (intensity) would have order $2n - 2 > 0$ in $1/r$. Since we are ultimately interested in the power radiated to infinity, we see that in the limit $r \to \infty$, all such terms will $n > 1$ will not contribute to the intensity.

Hence, $\mathrm{d}P = \frac{r^2}{\mu_0 c}|\mathbf{E}_{rad}|^2\mathrm{d}\Omega = \frac{cr^2}{\mu_0}|\mathbf{B}_{rad}|^2\mathrm{d}\Omega$.

Now, going back to the dipole situation, we have computed that $\int \mathrm{d}V'\mathbf{j} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_{ret}$. It then follows from the above that

$$\mathbf{A}_{rad} = \frac{\mu_0}{4\pi r}\frac{\mathrm{d}\mathbf{p}_{ret}}{\mathrm{d}t}, \tag{10.3.12}$$

and hence we have the magnetic field

$$\mathbf{B}_{rad} = -\frac{\mu_0}{4\pi cr}\hat{\mathbf{r}} \times \frac{\mathrm{d}^2\mathbf{p}_{red}}{\mathrm{d}t^2} = -\frac{\mu_0}{4\pi cr}\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{ret}. \tag{10.3.13}$$

From here we just need to substitute the expressions to find

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0}{(4\pi)^2 c}|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{ret}|^2, \tag{10.3.14}$$

which is exactly the canned formula from the beginning of this section. So, we see that the coefficient out front has a factor of $1/(4\pi)^2$ from the Green function and the factor of $\frac{c}{\mu_0}\frac{\mu_0^2}{c^2}$ comes from the coefficient of the Poynting vector times the μ_0^2 again from the square of the Green function and the $1/c^2$ from the square of the derivative of the retarded time.

Anyway, to proceed with the problem at hand, we just need to plug the dipole moment we found in part (a) into this formula to get the radiated power per solid angle. So, we take the two derivatives of the dipole moment to find

$$\ddot{\mathbf{p}}_{ret} = -\frac{4\pi\chi\epsilon_0}{3+\chi}\omega^2 a^3 \mathbf{E}_i. \tag{10.3.15}$$

Then if we denote the direction of the incident electric field vector by $\hat{\mathbf{e}}$,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0}{16\pi^2 c}\left(\frac{4\pi\chi\epsilon_0}{3+\chi}\omega^2 a^3\right)^2 E_0^2|\hat{\mathbf{r}} \times \hat{\mathbf{e}}|^2 = \alpha|\hat{\mathbf{r}} \times \hat{\mathbf{e}}|^2, \tag{10.3.16}$$

where we have defined the prefactor α for brevity.

However, this is only the radiated intensity for a single polarization. The problem states that the incident light is unpolarized, so we have to first average over all of the possible polarizations. To do so, let us fix the direction of propagation of the incident wave to be $\hat{\mathbf{k}}_0 = \hat{\mathbf{z}}$. Then the electric field for this incident wave may point in any direction in the xy-plane, $\hat{\mathbf{e}} = \sin\beta\hat{\mathbf{x}} + \cos\beta\hat{\mathbf{y}}$ for any angle β, which we will be averaging over. The angular position we are looking at is arbitrary, and so we may write the unit vector $\hat{\mathbf{r}}$ as $\hat{\mathbf{r}} = \cos\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\phi\hat{\mathbf{z}}$. Computing the cross product, we find

$$\hat{\mathbf{r}} \times \hat{\mathbf{e}} = -\hat{r}_z cos\beta\hat{\mathbf{x}} + \hat{r}_z\sin\beta\hat{\mathbf{y}} + (\hat{r}_x\cos\beta - \hat{r}_y\sin\beta)\hat{\mathbf{z}}, \tag{10.3.17}$$

which squares to

$$|\hat{\mathbf{r}} \times \hat{\mathbf{e}}|^2 = \hat{r}_z^2 + (\hat{r}_x \cos\beta - \hat{r}_y \sin\beta)^2. \tag{10.3.18}$$

But now since $\sin\beta$ and $\cos\beta$ are orthogonal and both $\sin^2\beta$ and $\cos^2\beta$ average to 1/2, we find that

$$\langle |\hat{\mathbf{r}} \times \hat{\mathbf{e}}|^2 \rangle = \frac{1}{2}(1 + \cos^2\phi), \tag{10.3.19}$$

where ϕ is the zenith angle, so the radiated power is

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{1}{2}\alpha(1 + \cos^2\phi) = \frac{\mu_0}{32\pi^2 c}\left(\frac{4\pi\chi\epsilon_0}{3+\chi}\omega^2 a^3\right)^2 E_0^2(1 + \cos^2\phi). \tag{10.3.20}$$

Our next task for this part of the problem is to compute the scattering cross-section for the problem. As we will see, however, almost all of the heavy lifting has already been done. So first, we must define the differential scattering cross-section, $\mathrm{d}\sigma$. Essentially, this is just another way of looking at the quantity $\mathrm{d}P$ in a slightly less situation-dependent way. Notice that the power (10.3.20) is proportional to the amplitude of the incident wave, and therefore proportional to the incident energy flux. It makes sense to normalize out the incident flux when considering the scattered power. So, we define

$$\mathrm{d}\sigma = \frac{\langle \mathrm{d}P \rangle}{|\langle \mathbf{S}_{inc} \rangle|} = \frac{\langle \mathrm{d}P/\mathrm{d}\Omega \rangle}{|\langle \mathbf{S}_{inc} \rangle|}\mathrm{d}\Omega. \tag{10.3.21}$$

So, for the sake of this problem, we only need that $|\langle \mathbf{S}_{inc} \rangle| = \frac{1}{2}\epsilon_0 c E_0^2$ and the power we just found to write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mu_0^2}{16\pi^2}\left(\frac{4\pi\chi\epsilon_0}{3+\chi}\omega^2 a^3\right)^2 (1 + \cos^2\phi) = \left(\frac{\chi}{3+\chi}\frac{\omega^2 a^3}{c^2}\right)^2 (1 + \cos^2\phi) \tag{10.3.22}$$

It is worth mentioning that there is a general result concerning what we've done here. As we mentioned above, we expanded the fields to order $1/r$ everywhere, which is just the largest term in the Laurent expansion which vanishes at infinity – we could call it the lowest order expansion of the fields about infinity. The total field in the far-field region is the sum of the incident wave and the scattered radiation, $\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{rad}$. Unless the system of interest is radiating in some way which is independent of the incident wave, which is typically not the case when dealing with scattering problems, we know that the magnitude of the radiation field will be proportional to the incident electric field's amplitude. It therefore makes sense to write $\mathbf{E}_{rad} = \frac{1}{r}E_0\mathbf{f}(\mathbf{k})e^{i(\mathbf{k}r - \omega t)}$ for some function $\mathbf{f}(\mathbf{k})$. Then the scattering cross-section may be written

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |\mathbf{f}(\mathbf{k})|^2, \tag{10.3.23}$$

which, we point out, makes contact with scattering theory in quantum mechanics.

10.4 Part (c)

For the case $a \sim \lambda$, we can no longer approximate the incident field as being constant across the sphere. As such, the potential theory calculation of part (a) would be worthless since it is dependent on there being a uniform incident wave and, in particular, the integral (10.2.9) would no longer be the integral of a constant as the polarization $\mathbf{P}$ would be spatially dependent.

To deal with this, we might be able to use the Born approximation, which approximates the field inside the dielectric, and therefore the field responsible for determining $\mathbf{P}$, as being equal to the incident electric field. That is, $\mathbf{P} \approx \epsilon_0 \chi \mathbf{E}_{inc}$. This approximation is, however, only good for weak dielectrics and weak conductors, so it would also require an assumption on the value of χ.

10.5 Part (d)

If we are considering the limit $a \gg \lambda$, we are essentially in the regime of geometrical optics. In this limit, we expect the the *total* cross section to converge to $2\sigma_{geom} = \pi a^2$.

The discussion of why this is the case is also a good opportunity to gain an intuition for what the cross-section is describing. Let us consider the case of a perfectly reflecting disk, oriented with normal along the direction of propagation for the incident wave in the short wavelength limit. In this limit, the disk casts a sharp circular shadow. As such, we would expect the amplitude of the scattered light to be zero everywhere except possibly where the light is reflected and, as we will see, in the shadow. So, when we integrate (10.3.23) over solid angle, only the cylinder defined by the disk will matter. However, if we look in the reflected portion, we see that $|\mathbf{f}(\mathbf{k})|^2$ must be unity here because all of the incident energy is reflected, and hence the amplitude of the reflected radiation must be equal to that of the incident wave. If we look in the shadow, we might not intuit that there is radiation here, however, the way we have defined things is only to consider the part of the total electric field which is not the incident wave. Well, in the shadow region, the *total* electric field is zero – which necessitates that the radiation field is the negative of the incident field so they cancel to create the shadow. Since the cross-section cares not for directionality, it follows that $|\mathbf{f}(\mathbf{k})|^2$ must be unity in this region as well. Hence, we actually have two regions of area σ_{geom} which add together to make $\sigma = 2\sigma_{geom}$.

Now, this result is explained for the specific case of a particular geometry and a perfectly reflecting object. There is a bit of a trick about how the cross-section is defined for a perfectly absorbing object, but the end result is that in the small wavelength limit, the cross-section converges to $2\sigma_{geom}$ there as well. In fact, it is conjectured that this convergence happens independent of absorption and geometry, as long as we are in the small wavelength limit. Numerical investigations appear to indicate that this is the case, though this remains an open question of electromagnetism since no major analytic results are known. This phenomena is known as the Extinction Paradox, though there is, of course, no paradox to be had.

11 Problem 10: Electromagnetism

11.1 Problem Statement (Special Relativity)

(a) An infinitely long straight wire of negligible cross-sectional area with a uniform linear charge density q_0 is at rest in the inertial frame K'. The frame K' moves with a speed $v = \beta c$, where c is the speed of light, along the axis of the wire with respect to the laboratory frame, K. Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory frame.

(b) What are the charge and current densities associated with the wire in its rest frame?

(c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part (a).

11.2 Part (a)

The first part of this problem is a simple matter of using Gauss' law to find the electric field. Since there is no current in the K' frame, there is no magnetic field. Using Gauss' law, the electric field is simply

$$\mathbf{E}' = \frac{q_0}{2\pi\epsilon_0 r}\hat{\boldsymbol{\rho}}. \tag{11.2.1}$$

So, we now need to boost the fields to the laboratory frame. Since the system K' moves with a velocity v in K, it follows that the boost from K' to K is by a velocity $-v$. Now, in theory, we could remember the form of the boost matrix, then apply it to the 4-potential, and then derive the transformation rules for the electromagnetic fields. Or, we could recall that the field strength matrix transforms like a Lorentz tensor and takes the form

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{11.2.2}$$

then apply the boost to both indices. However, these are both sub-optimal solutions for the sake of a timed exam, so it is likely better just to remember that the transformation law on the fields for boost velocity $\mathbf{v}$ from an unprimed frame to a primed frame is

$$\begin{aligned} \mathbf{E}'_{||} &= \mathbf{E}_{||}, & \mathbf{B}'_{||} &= \mathbf{B}_{||}, \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}), & \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B}_{\perp} - \tfrac{\mathbf{v}}{c^2} \times \mathbf{E}_{\perp}\right). \end{aligned} \tag{11.2.3}$$

So, for the problem at hand, the laboratory fields are

$$\mathbf{E} = \gamma \frac{q_0}{2\pi\epsilon_0 r}\hat{\boldsymbol{\rho}}, \qquad \mathbf{B} = \gamma \frac{v}{c^2}\frac{q_0}{2\pi\epsilon_0 r}\hat{\boldsymbol{\theta}}, \tag{11.2.4}$$

where $\hat{\boldsymbol{\theta}}$ is the angular direction in cylindrical coordinates. We also note that since the distance r and the direction $\hat{\boldsymbol{\rho}}$ are orthogonal to the boost direction, they are the same in both frames, and so we don't need to worry about transforming the coordinates.

11.3 Part (b)

Since $j^\mu = (\rho, \mathbf{j})$ is a vector which transforms under the Lorentz group, it could be feasible to remember the boost matrix and apply it to j^μ. Regardless though, the result is

$$\rho' = \gamma\rho, \qquad \mathbf{j}'_{\perp} = \mathbf{j}_{\perp}, \qquad \mathbf{j}'_{||} = \gamma(\mathbf{j}_{||} - \rho\mathbf{v}), \tag{11.3.1}$$

where we are again going from the unprimed to the primed frame with boost velocity $\mathbf{v}$. So, for our case,

$$\rho = \gamma\rho', \qquad \mathbf{j}_{||} = \gamma\rho'\mathbf{v} = \gamma\rho' v\hat{\mathbf{z}}. \tag{11.3.2}$$

11.4 Part (c)

Finally, we just need to compute the electromagnetic fields in the laboratory frame using the charge and current densities just found and check that they match the Lorentz boosted fields we found in part (a). So, we apply Gauss' and Amperé's laws once again to write

$$\mathbf{E} = \frac{\gamma q_0}{2\pi\epsilon_0 r}\hat{\boldsymbol{\rho}}, \qquad \frac{\mu_0 \gamma v q_0}{2\pi r}\hat{\boldsymbol{\theta}} = \frac{v}{c^2}\frac{\gamma q_0}{2\pi\epsilon_0 r}\hat{\boldsymbol{\theta}}, \tag{11.4.1}$$

which match the fields found previously, as desired.

12 Problem 11: Statistical Mechanics

12.1 Problem Statement (Partition Functions)

Consider a gas of classical and non-interacting atoms in thermal equilibrium at temperature T in a container of volume V and surface area A. Each atom in the 3D bulk has zero potential energy, but when absorbed on the surface, has a negative potential energy $-E_0$ and can be treated as a 2D ideal gas. Each atom has mass m.

(a) Find the free energy F_B and chemical potential μ_B for the bulk gas with N_B atoms.

(b) Find the free energy F_S and chemical potential μ_S for the surface gas with N_S atoms.

(c) Compute the surface density $\sigma(\rho, T) = N_S/A$ in terms of the bulk density $\rho = N_B/V$ and T. What is the value of σ in the limit $\hbar \to 0$. (Hint: $N! \approx N^N/e^N$)

12.2 Part (a)

Now, before we get into just immediately calculating things, we should first discuss the usage of the partition function vs the grand partition function. At first glance, it may seem more reasonable to use the grand partition function to model the bulk and the surface separately, and so perhaps we should use the grand partition function for this problem. However, we could take the problem at face value and assume fixed N_B and N_S because the problem says so in the prompt, but we could also come to the conclusion that this is the reasonable thing to do ourselves once we realize that we should not, in fact, be trying to model the two system separately. Instead, if we consider the composite system, then it's clear that we should use the partition function since the total particle number is conserved. Then since the system are non-interacting, the total partition function may be written as the product of the partition functions of the two system separately, $Z_{tot} = Z_B Z_S$. This can also be seen by writing the Hamiltonian for the composite system and noting that it factors. Since the total particle number of the composite system is conserved, it follows that $N = N_B + N_S$ is not a variable of the the sample space we are on, but merely a parameter determining the *fixed* sample space. Therefore, we cannot vary the total N, and we see

$$\mu_{tot} \equiv -kT \frac{\partial}{\partial N} \ln Z_{tot} = 0. \tag{12.2.1}$$

However, we could also write this as

$$0 = -kT \frac{\partial}{\partial N} (\ln Z_B + \ln Z_S) = -kT \frac{\partial \ln Z_B}{\partial N_B} + kT \frac{\partial Z_S}{\partial N_S} = \mu_B - \mu_S, \tag{12.2.2}$$

and hence $\mu_B = \mu_S$ in order for the particle number of the composite system to be conserved. We could have also assumed this because it is usually the condition referred to as "equilibrium," but as we see, we could have deduced this ourselves.

In any case, we turn to the problem of computing the partition function for the bulk so we can actually finish this problem. The Hamiltonian in the bulk is simply $H_B = \sum_k \frac{1}{2}mv_k^2$. This obviously factors, and the particles are considered to be identical, so the partition function for the bulk is given in terms of the single particle partition function in the bulk by $Z_B = \frac{1}{N_B!}Z_1^{N_B}$. The single particle partition function is computed to be

$$Z_1 = \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{h^3}e^{-\beta p^2/2m} = \frac{V}{h^3}\int \mathrm{d}^3pe^{-\frac{\beta}{2m}p^2} = V\left(\frac{2m\pi}{h^2\beta}\right)^{3/2} = \frac{V}{\lambda^3} \tag{12.2.3}$$

where $\lambda = \sqrt{\frac{2\pi\hbar^2\beta}{m}}$ is a stupid meme known as the thermal wavelength. In any case, the bulk partition function is now

$$Z_B = \frac{1}{N_B!}\frac{V^{N_B}}{\lambda^{3N_B}} \approx \left(\frac{e}{N_B}\frac{V}{\lambda^3}\right)^{N_B}. \tag{12.2.4}$$

Alright, at this point, we recall the definition of the free energy as $F_B = -kT\ln Z_B$, so we may write it in the form

$$F_B = -kTN_B\left[1 - \ln N_B + \ln\frac{V}{\lambda^3}\right]. \tag{12.2.5}$$

Now, while F_B is a Legendre transformation of E_B, the pair (μ_B, N_B) is not transformed, hence $\frac{\partial F_B}{\partial N_B} = \mu_B$. So,

$$\mu_B = kT\ln\frac{N_B\lambda^3}{V}. \tag{12.2.6}$$

12.3 Part (b)

We now do the same thing for the surface subsystem. Very little changes between these two computations, through we do have to be careful to replace the appropriate powers of 3 with powers of 2 and remember to track the potential energy $-E_0$ around. As before, $Z_S = \frac{1}{N_S!}Z_1^{N_S}$ and

$$Z_1 = \int \frac{\mathrm{d}^2p\mathrm{d}^2q}{h^2}e^{-\beta p^2/2m+\beta E_0} = \frac{A}{h^2}e^{\beta E_0}\int \mathrm{d}^2pe^{-\frac{\beta}{2m}p^2} = \frac{A}{\lambda^2}e^{\beta E_0}, \tag{12.3.1}$$

so the full partition function is

$$Z_S = \frac{1}{N_S!} Z_1^{N_S} = \left(\frac{e}{N_S} \frac{A}{\lambda^2} e^{\beta E_0} \right)^{N_S} . \tag{12.3.2}$$

The definition of the free energy does not depend on dimension, so we use the same free energy formula to find

$$F_S = -kTN_S \left[1 - \ln N_S + \ln \frac{A}{\lambda^2} + \beta E_0 \right] , \tag{12.3.3}$$

which then gives a chemical potential

$$\mu_S = kT \ln \frac{N_S \lambda^2}{A} - E_0. \tag{12.3.4}$$

And that's all folks.

12.4 Part (c)

Now for this part, we are supposed to compute $\sigma = N_S/A$ as a function of $\rho = N_B/V$. This requires us to have some relation connecting the two system. Fortunately, we already know of such a connection which is forced on us by the conservation of the total particle number, and hence which we might expect to give a relationship between the number of particles in each of the two systems: $\mu_S = \mu_B$. In terms of the expressions just obtained, this imposes

$$kT \ln \frac{N_B \lambda^3}{V} = kT \ln \frac{N_S \lambda^2}{A} - E_0. \tag{12.4.1}$$

Fortunately for us, this relation is already in terms of the quantities σ and ρ. Solving this equation for σ, we find

$$\sigma = \lambda \rho e^{\beta E_0}. \tag{12.4.2}$$

Finally, we are asked about the limit $\hbar \to 0$. We note that the only quantity here which depends on $\hbar$ is the thermal wavelength $\lambda \propto \hbar$. Hence, in the $\hbar \to 0$ limit, the surface density goes to zero and all particles of the system accumulate in the bulk.